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MINIMAL SURFACES APPLICABLE TO SURFACES OF REVOLUTION.

By J. K. Whittemore.

§ 1. **Introduction.** One of the simplest and most interesting classes of minimal surfaces is composed of those which are applicable to surfaces of revolution. The treatises on differential geometry give the determination of these surfaces and some of their more obvious properties. There have appeared since their first determination in 1862 several papers in mathematical journals dealing with these surfaces. It is the purpose of this article in Part I to give an account of these papers,* together with a new theorem regarding conjugate systems, and in Part II to prove certain theorems concerning them, believed to be new, relating to their symmetries and to their double lines, and finally to state without proof a new property characteristic of these surfaces.

The Enneper-Weierstrass equations of a real minimal surface are

$$x = \frac{1}{2}\int (1 - u^2)F(u)du + \frac{1}{2}\int (1 - v^2)\phi(v)dv,$$

$$y = \frac{i}{2}\int (1 + u^2)F(u)du - \frac{i}{2}\int (1 + v^2)\phi(v)dv,$$

$$z = \int uF(u)du + \int v\phi(v)dv,$$

where u, v are conjugate variables, and $F(u)$, $\phi(v)$ are conjugate analytic functions. Minimal surfaces applicable to surfaces of revolution are given by

$$F(u) = Cu^{m-2},$$

where m is real. We shall denote these surfaces, following Haag, as B surfaces, or B_m if it is desired to emphasize the value of m; this notation is suggested by the fact that the surfaces were first determined by E. Bour.

If $m \neq 0$ we may without restriction choose $C = 1$ in $F(u)$. It may be shown that m and $-m$ give the same surface, so that we may suppose $m \geqq 0$. Special surfaces of particular interest are $m = 0$, $C = 1$, the catenoid; $m = 0$, $C = i$, the right helicoid; $m = 2$, Enneper's surface.

* Exact references to the treatises and papers considered are given in § 2 of this article.

Excluding the cases $m = 0$ and $m = 1$, the equations of B_m are

$$x = \frac{1}{2}\left(\frac{u^{m-1}}{m-1} - \frac{u^{m+1}}{m+1}\right) + \frac{1}{2}\left(\frac{v^{m-1}}{m-1} - \frac{v^{m+1}}{m+1}\right),$$

$$y = \frac{i}{2}\left(\frac{u^{m-1}}{m-1} + \frac{u^{m+1}}{m+1}\right) - \frac{i}{2}\left(\frac{v^{m-1}}{m-1} + \frac{v^{m+1}}{m+1}\right),$$

$$z = \frac{u^m + v^m}{m}.$$

The direction cosines of the normal are

$$X = \frac{u+v}{uv+1}, \qquad Y = \frac{i(v-u)}{uv+1}, \qquad Z = \frac{uv-1}{uv+1}.$$

The first and second fundamental forms are respectively

$$ds^2 = (uv+1)^2(uv)^{m-2}dudv, \qquad \Sigma X d^2x = -u^{m-2}du^2 - v^{m-2}dv^2.$$

The differential equation of the lines of curvature and of the asymptotic lines are respectively

$$u^{m-2}du^2 - v^{m-2}dv^2 = 0, \qquad u^{m-2}du^2 + v^{m-2}dv^2 = 0.$$

The total curvature is

$$K = \frac{-4}{(uv+1)^4(uv)^{m-2}}.$$

We shall frequently use the real variables, r and φ, where $u = re^{\phi i}$ and $v = re^{-\phi i}$. With these variables the equations are

$$x = \frac{r^{m-1}}{m-1}\cos(m-1)\varphi - \frac{r^{m+1}}{m+1}\cos(m+1)\varphi,$$

$$y = -\frac{r^{m-1}}{m-1}\sin(m-1)\varphi - \frac{r^{m+1}}{m+1}\sin(m+1)\varphi,$$

$$z = \frac{2r^m}{m}\cos m\varphi,$$

$$X = \frac{2r\cos\varphi}{r^2+1}, \qquad Y = \frac{2r\sin\varphi}{r^2+1}, \qquad Z = \frac{r^2-1}{r^2+1}, \qquad K = \frac{-4}{(r^2+1)^4 r^{2m-4}}.$$

Both sets of expressions for X, Y, Z hold for every minimal surface, and the curves (r), that is r = constant, are the Minding parallels of the surface, curves along which the plane tangent to the surface is everywhere equally inclined to the xy plane; the curves (φ) are the Minding meridians, curves along which the normal to the surface is parallel to a fixed plane containing the Z axis. Much of the interest of the surfaces B is centered

on the curves (r) and (φ). For any minimal surface their spherical representations are the meridians and parallels of the sphere; since for B K is constant along (r), these curves correspond to the parallels of the applicable surface of revolution; the curves (φ) are the orthogonal trajectories of (r), therefore correspond to the meridians of the surface of revolution, and are in consequence geodesic lines of B.

I.

§ 2. **Literature. Papers of Bour and Schwarz. Theorem on Conjugate Systems.** A discussion of the surfaces B is given by Darboux, Théorie des Surfaces, vol. I (2d ed.), pp. 367, 368, 392–395, with particular consideration of the "minimal helicoids," $m = 0$, p. 359, and of Enneper's surface, pp. 372–376; the Minding parallels and meridians are explained on p. 357. Shorter treatments are given by Eisenhart, Differential Geometry, pp. 328–330, 260, 267, 269, and by Bianchi, Lezioni di Geometria Differenziale, vol. II (2d ed.), pp. 321–324.

The following papers deal wholly or in part with surfaces B:

E. Bour, "Théorie de la déformation des surfaces." Journal de l'École Polytechnique, cahier 39 (1862), pp. 99–109.

H. A. Schwarz, "Miscellen aus dem Gebiete der Minimalflächen." Journal de Crelle, vol. 80 (1875), p. 295, published also in Gesammelte Mathematische Abhandlungen.

A. Ribaucour, "Étude sur les élassoides ou surfaces à courbure moyenne nulle." Mémoires Couronnés de l'Académie Royale de Belgique, vol. XLIV (1882), chapter XX, pp. 215–224.

A Demoulin, Bulletin des Sciences Mathematiques (2), vol. XXI (1897), pp. 244–252.

J. Haag, same Bulletin (2), vol. XXX (1906), pp. 75–94, also pp. 293–296.

E. Stübler, Mathematische Annalen, vol. 75 (1914), pp. 148–176.

Only small parts of the first three papers cited deal with the surfaces B. Demoulin's paper is chiefly and Haag's two papers are wholly devoted to them; Stübler's paper is occupied chiefly with related surfaces. We consider these papers in the order given.

Bour determines first the surfaces of revolution applicable to minimal surfaces. These surfaces of revolution are of little interest. He then determines the minimal surfaces corresponding. As previously stated, this is the first determination. He gives their equations with the coördinates r and φ. The Enneper-Weierstrass equations resulted from work of Enneper published in 1864 and of Weierstrass published in 1866, so were

not available to Bour. The only properties of these minimal surfaces given by Bour are contained in the following three theorems, stated by him without proof: (1) Every geodesic (φ) cuts at a constant angle all curves of each family of the lines of curvature, of each family of asymptotic lines, and of the family of level curves (z); (2) at every point the lines of curvature bisect the angles of the curves (z) and (r), and the asymptotic lines bisect the angles of the curves (z) and (φ); (3) the developable tangent to B along every curve (φ) is a cylinder whose elements are parallel to the xy plane. All of these theorems may be easily verified, but only the first is of interest for, as appears from a general theorem which we state and prove in the next paragraph, the second theorem is true for all minimal surfaces while the third theorem is true for all non-developable surfaces if in the statement "Minding meridian" be substituted for "curve (φ)."

The theorem, which we believe to be new, is as follows: On every surface, not developable, the level curves (z) and the Minding meridians (Y/X) form a conjugate system; the orthogonal trajectories of the level curves, sometimes called the curves of steepest ascent, and the Minding parallels (Z) also form a conjugate system.

Both parts of this theorem may be easily proved geometrically; the first part indeed is a limiting case of Koenig's well-known theorem, which is stated by Darboux* as follows: The sections of a surface made by the planes through a straight line D form a conjugate system with the curves of contact with the surface of cones whose vertices lie on D. If we suppose D to lie in the xy plane and to recede to infinity without changing its direction the first part of our theorem is obtained. The two parts may easily be proved analytically. The proof follows.

The condition for conjugate directions is, with the usual notation, $\Sigma dx\delta X = 0$, where d and δ are differentials in the two directions. For a level curve $dz = 0$, and for a Minding meridian $\delta(Y/X) = 0$, from which we have

$$\frac{\delta X}{X} = \frac{\delta Y}{Y}.$$

From the equation $\Sigma X dx = 0$ we have, if the surface is not developable, substituting $dz = 0$ and replacing X and Y by δX and δY, $\Sigma dx\delta X = 0$. To prove the second part of the theorem we denote by d_1 and δ_1 the differentials along the curve of steepest ascent and the Minding parallel respectively, so that

$$\Sigma dx d_1 x = dx d_1 x + dy d_1 y = 0, \qquad \delta_1 Z = 0.$$

* Vol. I (2d ed.), p. 164.

We have to prove

$$\Sigma d_1 x \delta_1 X = d_1 x \delta_1 X + d_1 y \delta_1 Y = 0.$$

The last equation is easily verified if we note that

$$dz = p dx + q dy = 0, \qquad \delta_1 X \delta_1 q - \delta_1 Y \delta_1 p = 0,$$

the second of these holding good since

$$\delta_1 Z = \frac{p\delta_1 p + q\delta_1 q}{(1 + p^2 + q^2)^{3/2}} = 0.$$

From the theorem just proved we may easily get the result obtained by Minding* in the paper in which are first considered the curves that we have called the Minding parallels and meridians. Minding proved that the only surfaces, not developable, on which these parallels and meridians form an orthogonal system are the molding surfaces, including surfaces of revolution, whose axis is the Z axis, and the minimal surfaces. We obtain these results at once: If the Minding parallels and meridians form an orthogonal system suppose first that at every point one of the directions d, d_1, δ, δ_1 coincides with that of a line of curvature at the point; then the four directions coincide with the directions of the two lines of curvature and the level curves (z) form a family of lines of curvature, and the surface is a molding surface whose axis is the Z axis.† If none of the four directions coincides with that of a line of curvature we may call the slopes of the tangents to the curves of differentials d, d_1, δ, δ_1 referred to the principal axes of the indicatrix at the point m, m_1, μ, μ_1 respectively, and have

$$m m_1 = -1, \qquad m\mu = m_1\mu_1 = e^2 - 1,$$

where e is the eccentricity of the indicatrix. Then the necessary and sufficient condition that the directions of δ and δ_1 be orthogonal is

$$\mu\mu_1 = -(e^2 - 1)^2 = -1$$

and $e^2 = 0$ or 2, so that the indicatrix is at every point a circle or a rectangular hyperbola, and the surface is a sphere or a minimal surface.

Returning to the consideration of the papers mentioned at the beginning of this section we find in Schwarz's Miscellen the proof that all real minimal surfaces applicable to surfaces of revolution are given by setting $F(u) = Cu^{m-2}$, where m is real, in the Enneper-Weierstrass equations. His rather brief proof is elaborated by Darboux.‡ Schwarz was the first to prove that any real minimal surface applicable to the minimal surface given by the Enneper-Weierstrass equations is obtained by substituting

* Journal de Crelle, vol. 44 (1852), p. 66.

† See Eisenhart, p. 308.

‡ Vol. I (2d ed.), pp. 392–395.

for $F(u)$ in these equations $e^{i\alpha}F(u)$, where α is a real constant.* The surfaces so obtained are now known as the *associate surfaces* of the given minimal surface. The value $\alpha = \pi/2$ gives the *adjoint surface*. G. Scheffers states† that K. Peterson had observed earlier that all associate minimal surfaces are applicable.

§ 3. **Papers of Ribaucour, Demoulin, Haag, and Stübler.** In Ribaucour's memoir a chapter is devoted to the surfaces B. Ribaucour states that this investigation was made at his request by Rouquet. We find here the first full study of the geometry of these surfaces. Throughout this work the values $m = 0$ or 1 must be excluded. The coördinates r, φ are used almost exclusively. No use is made of the Enneper-Weierstrass equations and Schwarz's determination $F(u) = Cu^{m-2}$. There appear all of Bour's results, some of them in different form. The new results of chief interest follow.

A surface B is the envelope of a family of cylinders, *similar* to each other, whose elements are parallel to the xy plane, and whose characteristics are the curves (φ).

A surface B_m is composed of congruent nappes, each containing a plane of symmetry—these planes are actually those of certain plane geodesics (φ) (see Darboux, loc. cit., p. 460)—obtained from one by rotation about the Z axis through successive angles $2\pi/m$. If $m = p/q$, where p and q are integers having no common factor, the number of congruent nappes is p. This number is m if m is an integer. If m is rational, B_m is algebraic. The length of an arc of a geodesic (φ) is determined. All surfaces associate to B are congruent to B. Ribaucour calls these associate surfaces "élassoides groupés," and proves many of their properties, but makes no reference to Peterson or Schwarz. It is not however proved here that all associate surfaces are obtained by rotating B about the Z axis. This fact was proved by Darboux, loc. cit., p. 395. Ribaucour falls into an error‡ in stating and attempting to prove that two applicable surfaces having parallel tangent planes at corresponding points must be *adjoint* minimal surfaces ("élassoides conjugués"). It is well known that any two *associate* minimal surfaces have this property, and Darboux has proved that there are no others—loc. cit., p. 383.

Ribaucour shows that if $m = p/q$ the class of B_m is $2q(p + q)$; if m is integral the degree is $(m + 1)^2$. Each curve (r) lies on a quadric of revolution:

$$x^2 + y^2 + \frac{m^2}{m^2 - 1} z^2 = \left(\frac{r^{m-1}}{m - 1} + \frac{r^{m+1}}{m + 1} \right)^2.$$

* "Bestimmung einer speciellen Minimalfläche," Ges. Math. Abh., vol. I, p. 119.

† Theorie der Flächen, vol. II (2d ed.), p. 360.

‡ P. 106.

Each geodesic (φ) lies on a cone of the second degree. It is stated and incorrectly proved that if m is integral the degree of the curves (φ) is $2m - 1$. The correct value is, as given by Haag, $m + 1$. Ribaucour's error may perhaps be understood if it is noticed that $2m - 1$ gives the correct value for the simplest case, $m = 2$, that is Enneper's surface. It is of interest to examine Ribaucour's incorrect proof, as it is an example of an error into which one may easily fall; the proof even when corrected is unsatisfactory. It is as follows:

Eliminating r from the equations of B_m we have, changing to the notation of this paper,

$$[y \cos (m-1)\varphi + x \sin (m-1)\varphi][y \cos (m+1)\varphi - x \sin (m+1)\varphi] = \frac{m^2 z^2 \sin^2 m\varphi}{m^2 - 1},$$

$$2[y \cos (m+1)\varphi - x \sin (m+1)\varphi]^m = \frac{(mz \sin m\varphi)^{m-1} \sin 2m\varphi}{(m-1)^m}.$$

The curve (φ) lies on each of the surfaces given by these two equations, where φ is regarded as constant, of degree 2 and m respectively, and is therefore their intersection, or a part of their intersection. Ribaucour notes that these surfaces have in common the straight line,

$$y \cos (m+1)\varphi - x \sin (m+1)\varphi = z = 0,$$

and asserts in consequence that the curve (φ) is of degree $2m - 1$. The straight line however is a multiple line of the second surface of multiplicity $m - 1$, and must be counted as $m - 1$ lines in the intersection of the two surfaces, so that the degree of the remaining part of the intersection is $2m - (m - 1) = m + 1$, which is indeed the correct value of the degree of (φ). But though this result is correct the proof is unsatisfactory, for we cannot be sure that the two surfaces have not other intersections in addition to the straight line and the curve (φ). The simplest and the only reliable method of determining the degree of (φ) is to regard the equations of the surface as parametric equations of (φ) with parameter r, from which it appears immediately that such a curve intersects the general plane in $m + 1$ points when m is integral.

Among the geodesics (φ) are certain plane curves, all congruent. Enneper's surface is of degree 9 and class 6; $B_{1/2}$ is of degree 12 and class 12.

The paper of Demoulin is divided into three parts. In the first is given a simple proof of a new property characteristic of surfaces B: The necessary and sufficient condition that a minimal surface be the middle envelope of a rectilinear congruence and that the developables of the

congruence correspond to the lines of curvature of the surface is that the latter be applicable to a surface of revolution. In this case the congruence is a " Congruence of Ribaucour " and the director surface is the adjoint minimal surface.* Demoulin's proof is based on an interesting theorem proved very easily by him with the aid of the moving trihedral:† In order that a surface be applicable to a surface of revolution it is necessary and sufficient that there exist in the tangent plane at each point M of the surface a point O such that the infinitesimal displacement of O corresponding to any infinitesimal displacement of M be parallel to the plane determined by the normal at M and the displacement of M.

The second part of the paper gives a determination of the functions $F(u)$ and $\phi(v)$ by means of identifying the linear element with the known form for a surface of revolution, $ds^2 = f(u - v)dudv$. The most interesting thing in Demoulin's paper is the finding of certain *imaginary* minimal surfaces applicable to surfaces of revolution, not previously known. These are given by the Enneper-Weierstrass equations where

$$F(u) = Ce^{mu}, \qquad \phi(v) = C_1 \frac{e^{m/v}}{v^4}.$$

That these imaginary surfaces were not discovered by the method of Schwarz and Darboux may be explained by remarking that their method is based on geometrical intuition which can be applied only to real surfaces.

In the last part of this paper the determination of minimal surfaces applicable to a spiral surface is made to depend in a simple way on the preceding work.

The two papers of Haag contain new and interesting results, and give evidence of much labor, but are open to some criticism.

The first paper is devoted to a new determination of the surfaces B, by means of the moving trihedral, and to a study of the properties of these surfaces by the same method and by the ordinary methods of differential geometry. We cannot avoid the feeling that much of this paper is rather an exercise in the employment of the method of the moving trihedral than a useful application of the method. The author proves various properties of the B surfaces previously given, as he states, by Bour, but seems not to be acquainted with the work of Ribaucour for he gives, apparently as new, several theorems contained in the latter's memoir. Among Haag's results are the following: The surfaces B_m and B_{-m} are the same; among the geodesics (φ) are certain straight lines; all geodesics (φ) and no other geodesics are helices; the helicoidal developables tangent to B along curves (r) cut the xy plane in hypocycloids, homothetic with

* See Eisenhart, pp. 420–422.

† Bulletin de la Société mathematique de France, vol. XXII (1894), p. 47.

respect to the origin; consequently the edge of regression of each of these developables is an evolute of a hypocycloid; the cusps of these hypocycloids lie on the straight lines (φ), their vertices on the traces on the xy plane of the planes of the plane geodesic lines of curvature (φ).

Concerning the helicoidal developable we remark that it is the envelope of a one-parameter family of planes equally inclined to a fixed plane; its edge of regression is a helix whose axis is perpendicular to the fixed plane; the orthogonal trajectories of the straight lines of the surface, that is the involutes of the edge of regression, are the plane sections of the surface parallel to the fixed plane; conversely the surface of tangents to any helix is a helicoidal developable. Evidently the developable tangent to any surface along a Minding parallel is a helicoidal developable. These are the surfaces called in German " Böschungsflächen."

For $r = 1$ this developable is a cylinder whose elements are parallel to the Z axis. Haag remarks that the apparent contour of B on the xy plane is therefore a hypocycloid, but even the simplest B surface, Enneper's surface, has no apparent contour on the xy plane as is at once evident from an examination of Brill's familiar model of that surface: all the elements of the cylinder, $r = 1$, cut the surface. We remark that for Enneper's surface the hypocycloids are four cusped. Haag does not observe that if $m < 1$ hypocycloid must in all these statements be replaced by *epicycloid*. He shows also that the cone tangent to B whose vertex is the origin is of revolution, and is real only if $m < 1$.

It is evident that the properties of B_m differ considerably in the two cases, m greater or less than one. We may here record one such property which seems not to have been previously noticed: The surface B_m admits a real geodesic (r) when and only when $m < 1$, namely,

$$r = \sqrt{\frac{1 - m}{1 + m}}.$$

Haag's second paper is devoted to the determination of the degree and class of B_m and of certain curves on this surface when the surface is algebraic, that is when $m = p/q$. Several of his results are contained in Ribaucour's paper. We find that certain of his new results are incorrect.

His determination of the class of B_m agrees with Ribaucour's. The degree of B_m is given by the formula

$$d = \mu^2 - \omega,$$

where μ is the degree of the minimal curve Γ whose translation generates the surface, and ω is the number of points at infinity in which Γ meets

its conjugate. The equations of Γ may be written, if we set $u^{1/q} = v$,

$$x = \frac{1}{2}\left(\frac{v^{p-q}}{m-1} - \frac{v^{p+q}}{m+1}\right), \qquad y = \frac{i}{2}\left(\frac{v^{p-q}}{m-1} + \frac{v^{p+q}}{m+1}\right), \qquad z = \frac{v^p}{m}.$$

If $m > 1$, that is if $p > q$, the degree of Γ is $p + q$; it is easily seen that in this case $\omega = 0$, and $d = (p + q)^2$, equal, when m is an integer, to $(m + 1)^2$, agreeing with Ribaucour. But if $m < 1$, that is if $p < q$, the degree of Γ is not $p + q$, as stated, but $2q$. Further Haag's determination of ω for this case seems incorrect; for the multiplicity of the point of Γ at infinity given by $v = \infty$ is q, and that of the point at infinity given by $v = 0$ is $q - p$, so that $\omega = 2q(q - p)$, not $q^2 - p^2$. Then d is not $2p(p + q)$, as given by Haag, but

$$d = 4q^2 - 2q(q - p) = 2q(p + q).$$

We note that the latter formula gives the degree of $B_{1/2}$ as 12, as stated by Ribaucour. It is remarkable that if $m < 1$ the class and degree of B_m are aways the same. The degree is always greater than the class if $m > 1$. The degree of the curves (r) is $2(p + q)$, of the geodesics (φ) $p + q$ if $m > 1$, $2q$ if $m < 1$. Haag gives the latter as $p + q$ in both cases.

Stübler's paper is very interesting, but is so compact that it seems impossible to give more than an unsatisfactory outline of its contents. We regret that he has not noticed the papers of Haag. Only a short section is given to the surfaces B, the greater part of the paper being devoted to the discussion of related surfaces.

The first part contains various theorems previously known and some new results, chiefly relating to the helicoidal developables tangent to B along the curves (r), in particular results connected with their development on a plane and certain related cycloidal curves. It appears from his work that the projections on the xy plane of the curves (r) are hypo- or epitrochoids. He gives a new property characteristic of the surfaces B, which is easily proved: The only real minimal surfaces having a real curve along which the total curvature is a maximum are the surfaces B_m, $|m| \leqq 2$; the curve is $r = \sqrt{\frac{2 - m}{2 + m}}$.

The second section contains a discussion of the surfaces of center, or evolute surfaces, of B. The most interesting result is that the curve on each nappe corresponding to the curve of maximum total curvature on B is a singular curve of constant curvature.

In the third section are considered those involute surfaces of B whose normals are the tangents of the geodesics (φ). These are W surfaces and

have one family of plane lines of curvature—this appears at once from the consideration that all involutes of a helix are plane. The lines of curvature of the second family are spherical. If $m = p/q$ the degree of these surfaces is $2q(p + q)$ equal to the class of B_m, and, if $m < 1$, to the degree of B_m. When $m > 1$ this degree is less than that of B_m.

In the fourth section the cases, $m = 1$ and 0, previously excluded are taken up, and their evolute and involute surfaces discussed.

In the second part of the paper Stübler determines all minimal surfaces having certain properties in common with the surfaces B. The first section of the second part, section five, is devoted to the determination of minimal surfaces having an isothermal system composed of two families of curves each one of which has constant geodesic curvature. The only surfaces of this sort, besides B, are given by the Enneper-Weierstrass equations, where

$$F(u) = \frac{Ck}{(k - u^2)\sqrt{(k - u^2)(1 - ku^2)}}.$$

These surfaces are studied with the aid of elliptic functions.

The sixth and last section is given to the determination of minimal surfaces which are the envelopes of a family of similar cylinders whose elements are parallel to the xy plane and which may be brought into similar positions by rotation about the Z axis. Surfaces are found for which $F(u)$ has three different forms each depending on several parameters, and include the spiral minimal surfaces given by $F(u) = Cu^{-2+ni}$, where n is real. We note a slip at the bottom of page 174, where Stübler speaks of spiral minimal surfaces as given, in the notation of this article, by $F(u) = Cu^{ni}$, omitting -2 from the exponent.

II.

§ 4. **Symmetries of B surfaces.** There are certain properties of minimal surfaces applicable to surfaces of revolution with regard to symmetry which seem of sufficient interest and simplicity to deserve mention. Haag states in his first paper that he has studied the symmetries of these surfaces, but he gives no results.

Ribaucour states that the surface B_m consists of congruent nappes, and that it is brought into coincidence with itself by a rotation about the Z axis through the angle $2\pi/m$. It may be proved that if B_m is rotated about the Z axis through the angle $\pi + (\pi/m)$ it coincides with its symmetry with respect to the xy plane, so that if $m = (2a + 1)/(2b + 1)$, where a and b are integers, there are, as will appear later, other rotations besides those given by Ribaucour which bring the surface into coincidence with itself. The consideration of this and of other symmetries seems necessary

for the proof that B_m and B_{-m} are identical and not only symmetrical surfaces, as also for the proof that all plane non-rectilinear geodesics (φ) are congruent and are p in number when $m = p/q$.

In order that B_m have symmetry with respect to the yz plane it is necessary and sufficient that $m = 2a/(2b + 1)$, where a and b are integers; symmetry with respect to the xy plane $m = (2a + 1)/(2b + 1)$; symmetry with respect to the origin $m = (2a + 1)/2b$; the surface is always symmetrical with respect to the xz plane since that plane contains a geodesic of the surface.

The sufficiency of these conditions is at once obvious on the substitution of $\pm u$, $\pm v$ for u, v in the equations of the surface. The necessity may be shown as follows: if, for example, B_m has symmetry with respect to the yz plane, the plane geodesic (φ), $u = v$, is symmetrical with respect to the Z axis. The equations of this curve are

$$x = \frac{r^{m-1}}{m-1} - \frac{r^{m+1}}{m+1}, \qquad y = 0, \qquad z = \frac{2r^m}{m}.$$

There must be a value r' giving $x' = -x$, $z' = z$. The last condition requires that $r = r'e^{2k\pi i/m}$, where k is an integer; the first that there be two integers, n and n', such that

$$2k\pi i\frac{m-1}{m} = (2n+1)\pi i, \qquad 2k\pi i\frac{m+1}{m} = (2n'+1)\pi i,$$

from which it follows that $m = 2a/(2b + 1)$. It may be proved that the straight lines (φ) lie in the planes of the plane geodesics (φ) if and only if $m = (2a + 1)/(2b + 1)$.

§ 5. **Multiple points and lines. Results and discussion of the case $r_1 = r$.** We shall prove in this and the following sections:

If $m > 1$, the plane of each plane geodesic (φ) of B_m contains a real double curve; all these double curves are congruent; the surface has no other real finite double curves; if $m = p/q$ there are p such curves; these curves are algebraic and of degree $\frac{1}{2}(p + q)(p + q - 1)$.

If $m < 1$, the plane of each plane geodesic (φ) contains a real double curve, and all these curves are congruent; if $m = p/q$ there are p such curves and their degree is $q(p + q - 1)$; if $m = p/q$ the straight lines φ, which lines are the intersection of B_m with the xy plane, are also double lines and are p in number.

If $m = 1$, the surface is periodic and contains an infinite number of congruent plane geodesics (φ) in equidistant planes parallel to the xz plane, and an infinite number of congruent plane double curves, one in each plane of a plane (φ). The xy plane is cut by B_1 in an infinite number of parallel straight lines (φ) bisecting the distances between these planes.

If $m = 0$, the surfaces are the minimal helicoids. The catenoid and the right helicoid have no real finite multiple points or lines. In all other cases the surface has an infinite number of double curves (r), which are circular helices, and along each of which two branches of the surface intersect at a constant angle.

In the determination of the real finite multiple points of the surfaces B_m we exclude at first the values $m = 1$ and $m = 0$. Writing, as before, $u = re^{\phi i}$ and $v = re^{-\phi i}$, we exclude $r = 0$, noting that this value gives

$$x = y = z = 0, \quad m > 1, \qquad x = y = \infty, \quad z = 0, \quad m < 1.$$

Real points of the surface are given in general by conjugate values of u, v or by real values of r and φ, and for such a point it is no restriction to suppose r positive. A *multiple point* is a point x, y, z given by more than one pair of values of u, v.

When x, y, z have fixed values, the same is true of the three expressions

$$x^2 + y^2 + \frac{m^2}{m^2 - 1} z^2 = \left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1} \right)^2, \qquad x + iy = -\frac{u^{m+1}}{m+1} + \frac{v^{m-1}}{m-1},$$

$$\frac{mz}{2} = r^m \cos m\varphi.$$

Conversely, if for more than one pair of values of u, v or r, φ these expressions have the same values, the coördinates x, y, z have the same values, if the latter are real and finite.

We enquire first if a point may be given by two pairs of coördinates, r, φ and r_1, φ_1, where $r_1 = r \neq 0$, $\varphi_1 \neq \varphi + 2n\pi$. From the value of z it follows $\cos m\varphi_1 = \cos m\varphi$, $\varphi_1 = (2k\pi/m) \pm \varphi$.

From $x + iy$, choosing $\varphi_1 = (2k\pi/m) - \varphi$,

$$-\frac{r^2}{m+1} e^{(m+1)[(2k\pi i/m) - \phi i]} + \frac{1}{m-1} e^{-(m-1)[(2k\pi i/m) - \phi i]}$$

$$= -\frac{r^2}{m+1} e^{(m+1)\phi i} + \frac{1}{m-1} e^{-(m-1)\phi i}.$$

This may be reduced to

$$\frac{r^2}{m+1} \sin\left[(m+1)\left(\varphi - \frac{k\pi}{m} \right) \right] + \frac{1}{m-1} \sin\left[(m-1)\left(\varphi - \frac{k\pi}{m} \right) \right] = 0.$$

This becomes, for $k = 0$,

$$\frac{r^2}{m+1} \sin (m+1)\varphi + \frac{1}{m-1} \sin (m-1)\varphi = 0.$$

This is exactly the equation, $y = 0$, from which $-r^{m-1}$ has been cancelled. Every point of B_m in this plane is given by two pairs of coördi-

for the proof that B_m and B_{-m} are identical and not only symmetrical surfaces, as also for the proof that all plane non-rectilinear geodesics (φ) are congruent and are p in number when $m = p/q$.

In order that B_m have symmetry with respect to the yz plane it is necessary and sufficient that $m = 2a/(2b + 1)$, where a and b are integers; symmetry with respect to the xy plane $m = (2a + 1)/(2b + 1)$; symmetry with respect to the origin $m = (2a + 1)/2b$; the surface is always symmetrical with respect to the xz plane since that plane contains a geodesic of the surface.

The sufficiency of these conditions is at once obvious on the substitution of $\pm u$, $\pm v$ for u, v in the equations of the surface. The necessity may be shown as follows: if, for example, B_m has symmetry with respect to the yz plane, the plane geodesic (φ), $u = v$, is symmetrical with respect to the Z axis. The equations of this curve are

$$x = \frac{r^{m-1}}{m-1} - \frac{r^{m+1}}{m+1}, \qquad y = 0, \qquad z = \frac{2r^m}{m}.$$

There must be a value r' giving $x' = -x$, $z' = z$. The last condition requires that $r = r'e^{2k\pi i/m}$, where k is an integer; the first that there be two integers, n and n', such that

$$2k\pi i\frac{m-1}{m} = (2n+1)\pi i, \qquad 2k\pi i\frac{m+1}{m} = (2n'+1)\pi i,$$

from which it follows that $m = 2a/(2b + 1)$. It may be proved that the straight lines (φ) lie in the planes of the plane geodesics (φ) if and only if $m = (2a + 1)/(2b + 1)$.

§ 5. **Multiple points and lines. Results and discussion of the case $r_1 = r$.** We shall prove in this and the following sections:

If $m > 1$, the plane of each plane geodesic (φ) of B_m contains a real double curve; all these double curves are congruent; the surface has no other real finite double curves; if $m = p/q$ there are p such curves; these curves are algebraic and of degree $\frac{1}{2}(p + q)(p + q - 1)$.

If $m < 1$, the plane of each plane geodesic (φ) contains a real double curve, and all these curves are congruent; if $m = p/q$ there are p such curves and their degree is $q(p + q - 1)$; if $m = p/q$ the straight lines φ, which lines are the intersection of B_m with the xy plane, are also double lines and are p in number.

If $m = 1$, the surface is periodic and contains an infinite number of congruent plane geodesics (φ) in equidistant planes parallel to the xz plane, and an infinite number of congruent plane double curves, one in each plane of a plane (φ). The xy plane is cut by B_1 in an infinite number of parallel straight lines (φ) bisecting the distances between these planes.

If $m = 0$, the surfaces are the minimal helicoids. The catenoid and the right helicoid have no real finite multiple points or lines. In all other cases the surface has an infinite number of double curves (r), which are circular helices, and along each of which two branches of the surface intersect at a constant angle.

In the determination of the real finite multiple points of the surfaces B_m we exclude at first the values $m = 1$ and $m = 0$. Writing, as before, $u = re^{\phi i}$ and $v = re^{-\phi i}$, we exclude $r = 0$, noting that this value gives

$$x = y = z = 0, \quad m > 1, \qquad x = y = \infty, \quad z = 0, \quad m < 1.$$

Real points of the surface are given in general by conjugate values of u, v or by real values of r and φ, and for such a point it is no restriction to suppose r positive. A *multiple point* is a point x, y, z given by more than one pair of values of u, v.

When x, y, z have fixed values, the same is true of the three expressions

$$x^2 + y^2 + \frac{m^2}{m^2-1} z^2 = \left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1} \right)^2, \qquad x + iy = -\frac{u^{m+1}}{m+1} + \frac{v^{m-1}}{m-1},$$

$$\frac{mz}{2} = r^m \cos m\varphi.$$

Conversely, if for more than one pair of values of u, v or r, φ these expressions have the same values, the coördinates x, y, z have the same values, if the latter are real and finite.

We enquire first if a point may be given by two pairs of coördinates, r, φ and r_1, φ_1, where $r_1 = r \neq 0$, $\varphi_1 \neq \varphi + 2n\pi$. From the value of z it follows $\cos m\varphi_1 = \cos m\varphi$, $\varphi_1 = (2k\pi/m) \pm \varphi$.

From $x + iy$, choosing $\varphi_1 = (2k\pi/m) - \varphi$,

$$-\frac{r^2}{m+1} e^{(m+1)[(2k\pi i/m) - \phi i]} + \frac{1}{m-1} e^{-(m-1)[(2k\pi i/m) - \phi i]} = -\frac{r^2}{m+1} e^{(m+1)\phi i} + \frac{1}{m-1} e^{-(m-1)\phi i}.$$

This may be reduced to

$$\frac{r^2}{m+1} \sin\left[(m+1)\left(\varphi - \frac{k\pi}{m} \right) \right] + \frac{1}{m-1} \sin\left[(m-1)\left(\varphi - \frac{k\pi}{m} \right) \right] = 0.$$

This becomes, for $k = 0$,

$$\frac{r^2}{m+1} \sin (m+1)\varphi + \frac{1}{m-1} \sin (m-1)\varphi = 0.$$

This is exactly the equation, $y = 0$, from which $-r^{m-1}$ has been cancelled. Every point of B_m in this plane is given by two pairs of coördi-

nates, r, φ and r, $-\varphi$, or by u, v and v, u; the xz plane cuts the surface in the plane geodesic line of curvature, $u = v$, which is not a double curve, the two pairs of coördinates being in this case identical, and in a double curve for which $u \neq v$. From the values of X, Y, Z it is clear that the normals to the two branches of the surface at a point of this curve make equal angles with the X and Z axes and supplementary angles with the Y axis. If $m = p/q$, the equation of the double curve is found by cancelling from the equation above $\sin(\varphi/q)$. When $m = p/q$, the degree of the double curve is one half the difference of the degree of the surface and that of the plane geodesic, $u = v$. The degree is

$$\tfrac{1}{2}[(p+q)^2 - (p+q)] = \tfrac{1}{2}(p+q)(p+q-1), \qquad m > 1,$$

$$\tfrac{1}{2}[2q(p+q) - 2q] = q(p+q-1), \qquad m < 1.$$

When $m = (2a+1)/(2b+1)$ the straight line (φ), in this case the X axis, is part of the double curve.

Since B_m is brought into coincidence with itself or a symmetrical surface by a rotation about the Z axis through angles π/m there is a congruent double curve in the plane of each plane geodesic (φ); all these double curves are given by $\varphi_1 = (2k\pi/m) - \varphi$. When $m = p/q$ the number of these double lines is p.

We now form a similar equation from $x + iy$, taking $\varphi_1 = (2k\pi/m) + \varphi$. From this equation the factor $1 - e^{2k\pi i/m}$ may be cancelled. It becomes

$$\frac{r^2}{m+1} e^{(m+1)\phi i} - \frac{1}{m-1} e^{-(m-1)\phi i} = 0.$$

This is the equation $x + iy = 0$; we rewrite it

$$r^2 = \frac{m+1}{m-1}(\cos 2m\varphi - i \sin 2m\varphi).$$

If r is real, $2m\varphi = n\pi$, where n is an integer even or odd as m is greater or less than 1.

Suppose $m > 1$, then

$$r = \sqrt{\frac{m+1}{m-1}}, \qquad \varphi = \frac{n'\pi}{m}, \qquad n = 2n', \qquad x = y = 0,$$

$$z = \pm \frac{2}{m}\left(\frac{m+1}{m-1}\right)^{m/2},$$

the upper or lower sign being chosen in z as n' is even or odd. These are the points in which the plane geodesics (φ) cut the Z axis. The point or points in which the geodesic (φ) lying in the xz plane cuts the Z axis are

given by

$$r = \sqrt{\frac{m+1}{m-1}}, \qquad \varphi = h\pi, \qquad h \text{ integral.}$$

We conclude that this geodesic passes through both points on the Z axis if $m = p/q$ and p is odd, in other cases through one of the points. The double curve in the xz plane goes through both points on the Z axis except for $m = 2$, Enneper's surface, in which case it goes through the point not on the geodesic (φ) in that plane.

Suppose $m < 1$, then

$$r = \sqrt{\frac{1+m}{1-m}}, \qquad \varphi = \frac{2n'+1}{m}\frac{\pi}{2}, \qquad n = 2n'+1, \qquad x = y = z = 0.$$

The geodesic (φ) of the xz plane does not contain the origin, hence every double line of those considered does pass through the origin. It is of some interest to consider

$$r = \sqrt{\frac{1+m}{1-m}}, \qquad \varphi = \frac{2n'+1}{m}\frac{\pi}{2}, \qquad m > 1,$$

$$r = \sqrt{\frac{m+1}{m-1}}, \qquad \varphi = \frac{n'\pi}{m}, \qquad m < 1.$$

These coördinates give to u and v values which are not conjugate, so that X, Y, Z are not all real. If such coördinates give real values to x, y, z that point must be considered an isolated singular point of a real curve whose equation is satisfied by such coördinates r, φ. The first pair gives the origin, which is accordingly an isolated singular point of each double curve. The second pair gives again the points in which the plane geodesics (φ) cut the Z axis, but they are real only if $m = 2a/(2b+1)$. In this case one lies on each plane geodesic (φ). Both lie on each double line unless $p = 2$; if $p = 2$ then, as in the case of Enneper's surface, there are but two plane geodesics (φ) and two double curves; each has as an isolated singular point that one of the two points which is not an isolated singular point of the plane geodesic (φ) in the same plane.

We consider again the case $m = (2a+1)/(2b+1)$, where the straight lines (φ) form part of the double curves. We have here, as in other particulars, a striking difference in the cases $m > 1$ and $m < 1$. The line in the xz plane, that is the X axis, is given by $\varphi = \pm(\pi/2)$; we find

$$y = z = 0, \qquad x = \left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1}\right)\cos(a+b)\pi.$$

If $m > 1$, one half-line issuing from the origin is given by positive real

values of r, the other half-line only by complex values of r. One half-line is therefore the intersection of two real branches of the surface while the other half-line is not. If $m < 1$, all points of the X axis are given by positive real values of r and two real branches of the surface intersect all along the line.

§ 6. **Double lines** $r_1 \neq r$. We now consider the possibility of multiple points for which $r_1 \neq r$. We must have

$$\left(\frac{{r_1}^{m-1}}{m-1} + \frac{{r_1}^{m+1}}{m+1}\right)^2 = \left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1}\right)^2.$$

If $m > 1$, this equation can be satisfied only by $r_1 = r$ or $r_1 = -r$. The first case has been considered; the second can be reduced to it, for if $u_1 = r_1 e^{\phi_1 i}$ and $v_1 = r_1 e^{-\phi_1 i}$, $r_1 = -r$, we may write $u_1 = re^{(\phi_1+\pi)i}$ and $v_1 = re^{-(\phi_1+\pi)i}$. In this case B_m has no other real finite multiple points or lines than those already given. If $m < 1$, the equation above for r_1 is always satisfied by one and only one value $r_1 \neq \pm r$. Supposing r_1 to have this value, we have, for rational values of m, for

$$m = \frac{2a}{2b+1}, \qquad \varphi = \frac{2n+1}{m}\frac{\pi}{2}, \qquad \varphi_1 = \varphi + (2b+1)\pi,$$

$$\frac{{r_1}^{m-1}}{m-1} + \frac{{r_1}^{m+1}}{m+1} = -\left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1}\right),$$

$z = 0$ and the same values for x and y, giving points on the straight lines (φ). Two real branches of the surface intersect along each of these lines. The normals to the two branches at a point of such a line are not generally equally inclined to any of the coördinate axes. When

$$m = \frac{2a+1}{2b}, \qquad \varphi = \frac{2n+1}{m}\frac{\pi}{2}, \qquad \varphi_1 = \varphi + 2b\pi,$$

$$\frac{{r_1}^{m-1}}{m-1} + \frac{{r_1}^{m+1}}{m+1} = -\left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1}\right),$$

give the same result. The straight lines (φ) are in this case also double lines all along which two real branches intersect, the normals to the two branches making generally different angles with all the axes.

When $m = (2a+1)/(2b+1)$, if $r_1 \neq \pm r$, there is no value of φ_1 such that the points on the straight lines (φ) are given by the two pairs of coördinates r, φ and r_1, φ_1, but these lines have already been proved to be double lines, each point of such a line being given by two pairs of coördinates, r, φ and r, $-\varphi$. In this case we have each point x, y, 0 on such a line given by r, $\varphi = (2n+1)/m\,(\pi/2)$, and the point $-x$, $-y$, 0, symmetrical with respect to the origin, given by $r_1 \neq \pm r$, $\varphi_1 = \varphi + (2b+1)\pi$.

Ribaucour showed that each curve (r) lies on a quadric of revolution. If $m < 1$, there are on each of these quadrics, hyperboloids of one sheet in this case, two curves, (r) and (r_1), $r_1 \neq \pm r$; if $m = 2a/(2b+1)$ or $(2a+1)/2b$ these curves intersect on the straight lines (φ); if $m = (2a+1)/(2b+1)$, the curves (r) and (r_1) meet the xy plane at points on the straight lines (φ) symmetrical with respect to the origin.

§ 7. **The surfaces B_1.** Writing $u = re^{\phi i}$ and $v = re^{-\phi i}$ the equations of B_1 are

$$x = \log r - \tfrac{1}{2}r^2 \cos 2\varphi, \qquad y = -\varphi - \tfrac{1}{2}r^2 \sin 2\varphi,$$

$$z = 2r \cos \varphi.$$

The surface is obviously periodic, one period being given by

$$2\pi > \varphi \geqq 0, \qquad 2\pi > -y \geqq 0.$$

The value $m = -1$ gives the same surface in a different position. The surface has an infinite number of congruent plane geodesic lines of curvature (φ) in equidistant planes, parallel to the xz plane; the equations are $u = v$, or $\varphi = -n\pi$, or

$$x = \log r - \tfrac{1}{2}r^2, \qquad y = n\pi, \qquad z = \pm 2r.$$

The xy plane cuts the surface in an infinite number of parallel straight lines (φ) whose equations are $u = -v$, or $\varphi = -(2n+1)(\pi/2)$, or

$$x = \log r \pm \tfrac{1}{2}r^2, \qquad y = (2n+1)\frac{\pi}{2}, \qquad z = 0.$$

The curves (r) lie on the parabolic cylinders

$$z^2 + 4x = 2(\log r^2 + r^2).$$

As for B_m in general the curves (φ) are helices on similar cylinders whose elements are parallel to the xy plane.

We seek the real finite double lines. Any point x, y, z is given by only one value of $r \neq 0$ since $z^2 + 4x = 2(\log r^2 + r^2)$. If φ_1 and φ give the same point, we have from the value of z

$$\cos \varphi_1 = \cos \varphi, \qquad \varphi_1 = -2n\pi \pm \varphi.$$

The value of y gives

$$2n\pi \mp \varphi \mp \tfrac{1}{2}r^2 \sin 2\varphi = -\varphi - \tfrac{1}{2}r^2 \sin 2\varphi.$$

The upper signs give $2n\pi = 0$, $\varphi_1 = \varphi$; the lower signs

$$y = -\varphi - \tfrac{1}{2}r^2 \sin 2\varphi = n\pi.$$

There are an infinite number of congruent double curves lying in the

planes $y = n\pi$. Choosing $n = 0$ the two pairs of coördinates of a point of this curve are r, φ and r, $-\varphi$, $\varphi \neq k\pi$. The normals to the two branches intersecting in this line make equal angles with the X and Z axes, supplementary angles with the Y axis.

§ 8. **The surfaces B_0.** We may without restriction write for the minimal helicoids, $m = 0$,

$$F(u) = -\frac{e^{ai}}{u^2}, \qquad \phi(v) = -\frac{e^{-ai}}{v^2}.$$

The equations of the surface are

$$x = r\cos(\varphi + \alpha) + \frac{1}{r}\cos(\varphi - \alpha), \qquad y = r\sin(\varphi + \alpha) + \frac{1}{r}\sin(\varphi + \alpha),$$

$$z = 2\varphi\sin\alpha - 2\log r\cos\alpha.$$

The value $\alpha = 0$ gives the catenoid; $\alpha = \pi/2$ the right helicoid. For all other surfaces we may suppose $\pi/2 > \alpha > 0$, since the surfaces corresponding to $\pi \pm \alpha$ are symmetrical to the surface α. The equations may also be conveniently written so as to involve only trigonometric functions by putting $r = \operatorname{ctn}(\rho/2)$,* or so as to involve trigonometric and hyperbolic functions by putting $\log r = \rho$.† Among the geodesics (φ) are no plane curves and no lines of curvature except for the catenoid, where they are the meridians, and no asymptotic lines except for the right helicoid, where they are the rulings. The curves (r) are lines of curvature and asymptotic lines in the same cases, and are plane only for the catenoid.

The curves (r) are circular helices lying on the cylinders

$$x^2 + y^2 = r^2 + \frac{1}{r^2} + 2\cos 2\alpha, \qquad \cos\gamma = \frac{2r\sin\alpha}{r^2 + 1},$$

where γ is the angle which the tangent makes with the Z axis. The helix $r = 1$ is on the cylinder of minimum radius

$$x^2 + y^2 = 4\cos^2\alpha$$

and has the maximum pitch, $\gamma = (\pi/2) - \alpha$. It is Stübler's curve of maximum total curvature, and the minimal helicoid is tangent to the circular cylinder along it, so that it is a geodesic of the helicoid. On each of these cylinders whose radius is greater than $2\cos\alpha$ there are, with certain exceptions to be given presently, two helices, (r) and $(1/r)$, which have the same pitch. The geodesics (φ) lie on equal hyperbolic cylinders whose elements are parallel to the Z axis, and whose envelope is the cylinder $x^2 + y^2 = 4\cos^2\alpha$. The eccentricity of the hyperbola directrices is

* See Stübler's paper, p. 166.

† Bianchi, loc. cit., p. 324.

sec α. As for B_m in general the geodesics (φ) are the curves of contact of the minimal helicoid and similar cylinders whose elements are parallel to the xy plane.

We seek the multiple lines of the surface. It is obvious, and it may also be easily shown from the equations, that the catenoid, $\alpha = 0$, and the right helicoid, $\alpha = \pi/2$, have no real finite double curve or point. The situation is different when α is between 0 and $\pi/2$.

Since $x^2 + y^2 = r^2 + (1/r^2) + 2\cos 2\alpha$ a point x, y, z can be given only by r and $r_1 = 1/r$. If r is fixed it follows from the value of z that the same is true of φ. If $r_1 = 1/r$ and φ_1 give the same point as r, φ we have

$$x = \frac{1}{r}\cos(\varphi_1 + \alpha) + r\cos(\varphi_1 - \alpha) = r\cos(\varphi + \alpha) + \frac{1}{r}\cos(\varphi - \alpha),$$

$$y = \frac{1}{r}\sin(\varphi_1 + \alpha) + r\sin(\varphi_1 - \alpha) = r\sin(\varphi + \alpha) + \frac{1}{r}\sin(\varphi - a),$$

$$z = 2\varphi_1 \sin\alpha + 2\log r \cdot \cos\alpha = 2\varphi\sin\alpha - 2\log r \cdot \cos\alpha.$$

The last equation gives

$$\varphi_1 - \varphi = -2\log r \cdot \operatorname{ctn}\alpha.$$

The first two are equivalent to the single equation

$$r^2 = \frac{\sin\left(\alpha + \frac{\varphi_1 - \varphi}{2}\right)}{\sin\left(\alpha - \frac{\varphi_1 - \varphi}{2}\right)}.$$

Then r must satisfy the equation

$$r^2 = \frac{\sin(\alpha - \log r \cdot \operatorname{ctn}\alpha)}{\sin(\alpha + \log r \cdot \operatorname{ctn}\alpha)}.$$

Conversely, if r is any solution of this equation the coördinates r, φ and $r_1 = 1/r$, $\varphi_1 = \varphi - 2\log r \cdot \operatorname{ctn}\alpha$ give the same values for x, y, z.

We consider the solutions of the equation for r. It is clear that $r = 1$ is a solution. This value, as already noted, gives the cylinder of minimum radius, which is tangent to the helicoid along this helix. The latter is not a double curve, for evidently $r_1 = r$ and $\varphi_1 = \varphi$. Further it is clear that if r is a solution of the equation $1/r$ is also a solution. If we write $\lambda = \log r \operatorname{ctn}\alpha$, the equation can be written

$$\tan\lambda + \tan\alpha \tanh(\lambda\tan\alpha) = 0.$$

It may be proved that this equation for λ has an infinite number of real

roots, one and only one λ_n for each integral value of n, and such that

$$n\pi > \lambda_n > n\pi - \alpha.$$

As n is increased $\lambda_n - (n\pi - \alpha)$ decreases and approaches zero with great rapidity, at least as rapidly as $e^{-2\pi n}$. The values $r = e^{\pm \lambda_n \tan \alpha}$ give the same helix, a double curve, and the circular cylinders on which these lie carry each only one helix. For every curve (r) the tangent plane along the curve makes a constant angle with the xy plane, these curves being the Minding parallels of the helicoid. The same thing is of course true for each of the branches of the surface intersecting in a double curve; it appears that at each point the normals to the two branches make supplementary angles with the Z axis, and moreover form a constant angle all along the curve.

§ 9. **A new property characteristic of B surfaces.** We state without proof a new property characteristic of minimal surfaces applicable to surfaces of revolution. It is planned to publish a proof of this property on another occasion.

Schwarz proved* that corresponding points, that is points corresponding to the same values of the parameters u, v, of a family of associate minimal surfaces lie on an ellipse. The characteristic property is this:

The necessary and sufficient condition that the envelope of a family of minimal surfaces associate to a given minimal surface be a part of the locus of the extremities of the principal axes of Schwarz's ellipses, with coincidence of corresponding points, is that the given minimal surface be applicable to a surface of revolution, that is a surface B_m, where m is not 1 or 0.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY, NEW HAVEN, CONN.,
December 13, 1916.

* Loc. cit., p. 295.

A REDUCTION OF CERTAIN DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By William Duncan MacMillan.

Let us suppose there is given the differential equation

$$\frac{d\xi}{dt} = f(\xi), \tag{1}$$

where $f(\xi)$ is an analytic function of ξ which does not contain t explicitly. We propose to study the development of the function ξ in the neighborhood of the point ξ_0, which may be an ordinary point, a pole, a zero, or a branch point of the function $f(\xi)$ but not an essential singularity. For this purpose we make the substitution $\xi - \xi_0 = x$. With this substitution the right member of (1) can be expanded in powers of x. The coefficient of the first term, x^n, can be taken equal to unity, for if it were not unity let us suppose it is $a_0 \neq 0$. After dividing the equation through by a_0 and changing the independent variable from t to τ by means of the relation $\tau = a_0 t$, the coefficient of x^n in the resulting equation will be equal to unity. The equation becomes

$$\frac{dx}{dt} = x^n + a_1 x^{n+1} + a_2 x^{n+2} + \cdots. \tag{2}$$

The constant n will be a positive or negative integer according as the point ξ_0 is a zero or a pole of $f(\xi)$; it will be zero if ξ_0 is an ordinary point. If ξ_0 is a branch point n will be a fraction, and there will be a different expansion for each branch. Choosing one of these branches a suitable transformation on x will result in a series the exponents of which are integers. For example, if $n = \frac{1}{2}$ the transformation $x^{1/2} = y$ will give a series in integral powers of y. The equation (2) can therefore be assumed to have integral exponents.

We seek now a substitution

$$x = y + b_2 y^2 + b_3 y^3 + \cdots \tag{3}$$

such that the resulting differential equation in y shall be algebraic. On substituting (3) in (2) and arranging according to powers of y it is found that

$$\frac{dy}{dt} = y^n + [(n-2)b_2 + c_2]y^{n+1} + [(n-3)b_3 + c_3]y^{n+2} + \cdots + [(n-k)b_k + c_k]y^{n+k-1} + \cdots, \tag{4}$$

where the b_j are the coefficients of (3) and the c_k are constants which are polynomials in the constants a_j and b_j for which the subscript $j < k$.

On equating to zero the coefficients of y^{n+1}, y^{n+2}, $\cdots$, it is found that

$$b_k = -\frac{c_k}{n-k}, \qquad k = 2, \cdots \infty. \tag{5}$$

Consequently if n is a negative integer, zero, or $+1$ the right member of (4) can be reduced to its first term, so that the differential equation in y is simply

$$y' = y^n; \tag{6}$$

and the transformation (3) which effects this reduction is unique. If n is zero or a negative integer, say $n = -m$, the solution of (6) is

$$y = [(m+1)(t-t_0)]^{1/m+1}. \tag{7}$$

Granting for the moment that the series (3) converges for values of y different from zero, it is seen that, if $m = 0$, the point ξ_0 is an ordinary point of $\xi(t)$; but if $m > 0$ the point ξ_0 is a branch-point of $\xi(t)$. In the first case ξ is expansible in integral powers of $(t - t_0)$ in the neighborhood of ξ_0, and in the latter case in fractional powers.

If the exponent $n = +1$ the solution of equation (6) is

$$y = ce^t. \tag{8}$$

In this case x is expansible as a power series in e^t which approaches zero as t approaches $-\infty$. Since the point $t = \infty$ is an essential singularity of the function e^t it is clear that $t = \infty$ is also an essential singularity of the function $x(t)$.

If the exponent $n > +1$ it is seen from (5) that the particular coefficient b_n cannot be determined so as to make vanish the coefficient of y^{2n-1} in equation (4). There is no difficulty however in determining all of the other b_j, and equation (4) becomes

$$y' = y^n + c_n y^{2n-1}. \tag{9}$$

The transformation (3) is uniquely determined aside from the fact that the coefficient b_n remains arbitrary. There are therefore a single infinity of substitutions which transform (3) into (9); and it will be observed that (9) does not depend upon b_n.

The integral of equation (9) is defined by (provided $c_n \neq 0$)

$$-\frac{1}{c_n y^{n-1}} + \log\left(1 + \frac{1}{c_n y^{n-1}}\right) = \frac{n-1}{c_n}(t - t_0), \tag{10}$$

which is expressed more simply by taking

$$1 + \frac{1}{c_n y^{n-1}} = W, \qquad 1 - \frac{(n-1)}{c_n}(t - t_0) = \tau. \tag{11}$$

The result is

$$W - \log W = \tau, \tag{12}$$

which defines W as a function of τ. Let us denote this by writing $W = L(\tau)$. The function $L(\tau)$ satisfies the differential equation

$$\frac{dL}{d\tau} = L' = \frac{L}{L-1}, \tag{13}$$

whence

$$LL' = L + L' \qquad \text{and} \qquad L' - 1 = \frac{1}{L-1}. \tag{14}$$

From (11) it is found that

$$c_n y^{n-1} = \frac{1}{W-1} = L' - 1; \tag{15}$$

so that

$$y = \left(\frac{L'-1}{c_n}\right)^{1/(n-1)}. \tag{16}$$

It appears therefore that if $n \geqq 2$ and $c_n \neq 0$ the solutions of (2) are always expressible as power series in $[(L'-1)/c_n]^{1/(n-1)}$. If c_n should be zero equation (9) becomes simply

$$y' = y^n,$$

the solution of which is

$$y = \frac{1}{[(n-1)(t_0 - t)]^{1/(n-1)}}. \tag{17}$$

It is seen from (15) that y vanishes only for $W = L(\tau) = \infty$, and from (12) it is clear that $L(\tau)$ becomes infinite only for τ equal to infinity which is an essential singularity. Hence the function $y(t)$ has an essential singularity at $t = \infty$ provided $c_n \neq 0$, and from (17) it is easy to see that if $c_n = 0$ the function $y(t)$ has merely a branch point at $t = \infty$.

Equation (12) shows that if W is negative the corresponding values of τ are complex. If W is positive but very small τ also is positive and large. As W increases to $+1$, τ decreases to $+1$; and as W increases from $+1$ to ∞, τ also increases from $+1$ to ∞. For purposes of computation in which y must be small the corresponding value of W must be large, and tables of the functions $L(\tau)$ and $L'(\tau)$ will be found convenient. Such tables are given and the properties of $L(\tau)$ are discussed in the following paper.

En resumé, if n is a negative integer $x = 0$ is a branch-point of the

function $x(t)$. If $n = 0$ it is an ordinary point, and if n is a positive integer it is an essential singularity, except when the constant c_n is zero in which case it is merely a branch-point.

Convergence of the series.

In order to see that the substitution (3) is a convergent one let us consider x as a function of y defined by the equation

$$\frac{dx}{x^n + a_1x^{n+1} + a_2x^{n+2} + \cdots} = \frac{dy}{y^n + \beta y^{2n-1}}, \tag{18}$$

where the constant β in the denominator of the second member is at our disposal.

If n is a negative integer or zero, $n = -m$, we can take $\beta = 0$ and expand the left member in ascending powers of x, thus:

$$[x^m + \alpha_1x^{m+1} + \cdots]dx = y^m dy.$$

Integrating we have

$$x^{m+1} + \frac{m+1}{m+2}\alpha_1x^{m+2} + \cdots = y^{m+1} + \text{const.}$$

Since y must vanish with x the constant of integration is zero. Taking now $x = y(1 + \xi)$ and dividing through by y^{m+1}, we find

$$(m+1)\xi + \frac{m+1}{m+2}\alpha_1 y + \cdots = 0,$$

which has a unique and convergent solution for ξ as a power series in y for all positive values of m.

If n is unity we again take $\beta = 0$ and expand (18) with the result

$$\left[\frac{1}{x} + \alpha_0 + \alpha_1x + \cdots\right]dx = \frac{1}{y}dy.$$

Integrating we find

$$\log\frac{x}{y} + \alpha_0x + \tfrac{1}{2}\alpha_1x^2 + \cdots = 0. \tag{19}$$

On taking $x = y(1 + \xi)$ it is found that (19) admits a unique solution for ξ as a power series in y vanishing with y.

If $n > +1$ we find, on expanding both members of (18), series which it is convenient to write in the form

$$\left[\frac{1}{x^n} + \frac{n-2}{n-1}\frac{\alpha_{-(n-2)}}{x^{n-1}} + \cdots + \frac{\alpha_0}{(n-1)x} + \frac{\alpha_1}{(n-1)} + \frac{2\alpha_2}{(n-1)}x + \cdots\right]dx$$

$$= \left[\frac{1}{y^n} - \frac{\beta}{y} + \beta^2y^{n-2} - \beta^3y^{2n-3} + \cdots\right]dy.$$

Integrating and multiplying through by $-(n-1)$ there results

$$\frac{1}{x^{n-1}} + \frac{\alpha_{-(n-2)}}{x^{n-2}} + \cdots - \alpha_0 \log x - \alpha_1 x - \alpha_2 x^2 - \cdots$$
$$= \gamma + \frac{1}{y^{n-1}} + (n-1)\beta \log y + \beta^2 y^{n-1} \cdots,$$

where γ is the constant of integration. If now we take $x = y(1+\xi)$ it is clear that the term in $\log y$ will disappear only if $\beta = -\alpha_0/(n-1)$; and if β is given this value and we multiply through by y^{n-1} we will have

$$\left(\frac{1}{(1+\xi)^{n-1}} - 1\right) + \frac{\alpha_{-(n-2)}y}{(1+\xi)^{n-2}} + \cdots - \alpha_0 y^{n-1} \log (1+\xi)$$
$$- \alpha_1 y^n (1+\xi) - \cdots] = \gamma y^{n-1} + \beta^2 y^{2n-2} + \cdots. \tag{20}$$

This equation admits a unique solution for ξ as a power series in integral powers of y. The solution contains the arbitrary constant γ which appears first in the coefficient of y^{n-1}. Whatever be the value of the constant γ* the solution will converge provided y is sufficiently small. Consequently x is expansible as a power series in y containing the arbitrary constant γ in the coefficients of y^n and higher powers. The series thus derived satisfies the same conditions as does the series (3); they are therefore identical with it and the series (3) converges.

* The series $x = y + b_2 y^2 + \cdots + \gamma y^n + \cdots$ when considered as a function of $(t - t_0)$ apparently has two arbitrary constants, viz., γ and t_0. These two constants however are equivalent to only one, for we have

$$\int \frac{dx}{x^n + \cdots} = \gamma + \int \frac{dy}{y^n + C_n y^{2n-1}}.$$

Differentiating both sides with respect to γ and remembering that y is not a function of γ, we find

$$\frac{\partial x/\partial \gamma}{x^n + \cdots} = 1; \qquad \therefore \frac{\partial x}{\partial \gamma} = x^n + \cdots = \frac{\partial x}{\partial t} = -\frac{\partial x}{\partial t_0}.$$

Hence

$$\frac{\partial x}{\partial \gamma} + \frac{\partial x}{\partial t_0} = 0.$$

THE FUNCTION $W = L(\tau)$ DEFINED AS THE INVERSE OF THE FUNCTION $\tau = W - \log W$.

By W. D. MacMillan.

In the preceding paper W has been defined as a function of τ by the relation $W - \log W = \tau$, and this has been designated by the notation $W = L(\tau)$. We propose here to study the nature of this function.

This situation is analogous to the one defined by Kepler's Equation, $M = E - e \sin E$, in the problem of two bodies, where E is desired as a function of M.

From the point of view of the theory of functions $W = L(\tau)$ is an infinitely many-valued function of τ, and likewise τ is an infinitely many-valued function of W. The branch-points of $L(\tau)$ are determined by the equations

$$\begin{aligned} W - \log W &= \tau, \\ 1 - \frac{1}{W} &= 0, \end{aligned} \tag{1}$$

from which we find $W = 1, \tau = 1 - 2n\pi i$; and since the second derivative does not vanish for $W = 1$ only two branches interchange at the branch points. For the purpose of examining the complex values of the function we will take

$$W = \rho e^{i\phi}, \qquad \tau = \alpha + i\beta,$$

which substituted in the first equation of (1) gives the two equations

$$\begin{aligned} \rho \cos \varphi - \log \rho &= \alpha, \\ \rho \sin \varphi - \varphi &= \beta. \end{aligned} \tag{2}$$

Let us suppose now that W moves around the origin in a circle in the W-plane; then the value of ρ is constant and equations (2) show that τ describes a trochoid in the τ-plane, the radius of the rolling circle being equal to unity, the τ-point being at a distance ρ from its center, and the axis of rolling being the line $\alpha = 1 - \log \rho$. When ρ is greater than unity the trochoid has loops, each of which encloses a branch-point.

In order to have a one-to-one correspondence between the values of W and τ we will construct a Riemann surface for each of the variables. The equation $W - \log W = \tau$ is already solved for τ as a function of W. It is clear that τ will be a uniform function of W upon a W-surface of

infinitely many sheets which wind about the origin, a single sheet being included in the range $2n\pi \leqq \varphi < 2(n+1)\pi$.

Let us consider the values of τ for values of W lying in the principal sheet, $\rho \cos \varphi - \log \rho = \alpha_1$, $\rho \sin \varphi - \varphi = \beta_1$. Suppose W describes a circle for which ρ is small and $0 \leqq \varphi < 2\pi$. The corresponding path of τ is very nearly a straight line parallel to the β-axis connecting the points $(\rho - \log \rho, 0)$ and $(\rho - \log \rho, -2\pi)$ but this last end point excluded. For larger values of ρ, this line, which in reality is a trochoid described by a point near the center of the rolling circle, rapidly approaches the β-axis and increases slightly in curvature. For ρ equal to unity the trochoid becomes a cycloid connecting the two branch-points $(1, 0)$ and $(1, -2\pi)$. Consequently the interior of a circle of unit radius in the W-plane maps into a strip in the τ-plane of width 2π bounded by and including the real axis from the point $(1, 0)$ to $(+\infty, 0)$ and bounded by and excluding the straight line $\beta = 2\pi$, from the point $(1, -2\pi)$ to $(+\infty, -2\pi)$ and the cycloid connecting the two points $(1, 0)$ and $(1, -2\pi)$.

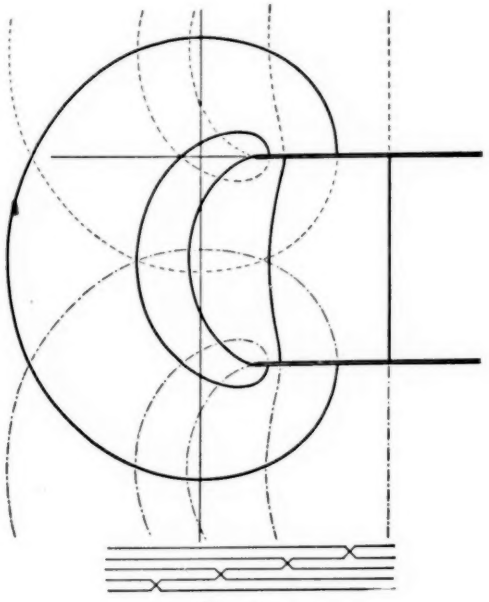

FIG. 1.

In order to avoid a double line in the τ-plane we will take for $\rho > 1$ the range $0 < \varphi \leqq 2\pi$. There a circle of radius slightly greater than unity maps into a trochoid of curvature greater than the cycloid connecting the points $(\rho - \log \rho, 0)$ and $(\rho - \log \rho, -2\pi)$, the first point excluded. As ρ increases the trochoid expands, the initial and end points receding

from the β axis. For very large values of ρ the trochoid becomes very nearly the arc of a circle connecting the points $(\rho - \log \rho, 0)$ and $(\rho - \log \rho, -2\pi)$, the radius of the circle being equal to ρ. Clearly the exterior of the unit circle in the principal W-sheet maps into the region bounded by and excluding the real axis from $(1, 0)$ to $(+\infty, 0)$ and the above-mentioned cycloid and bounded by and including the straight line $\beta = -2\pi$ between the points $(1, -2\pi)$ and $(+\infty, -2\pi)$, lying on the outside of these boundaries. Thus the principal sheet of the W-surface maps into an entire sheet in the τ-surface. The two straight lines described in the above boundaries are lines of discontinuity, that is, as τ crosses these lines the corresponding values of W change abruptly. Consequently we will suppose the τ sheet to be cut along these lines.

Consider now the values of τ corresponding to the values of W in the second sheet, for which $2\pi \leqq \varphi < 4\pi$. They are defined by the equations

$$\rho \cos \varphi - \log \rho = \alpha_2,$$

$$\rho \sin \varphi - \varphi \quad = \beta_2 + 2\pi.$$

Thus we have for the same values of ρ and φ, $\alpha_1 = \alpha_2$, $\beta_1 - 2\pi = \beta_2$, and the second τ-sheet differs from the first only in that the β coördinates are diminished by 2π. Let similar cuts be made in the second τ-sheet from the points $(1, -2\pi)$ and $(1, -4\pi)$ parallel to the real axis to $+\infty$. If the second τ-sheet be superposed upon the first the two sheets can be joined along the cuts proceeding from the points $(1, -2\pi)$, the lower edge in the first sheet being joined to the upper edge in the second sheet and the upper edge of first sheet to the lower edge of the second. If now τ crosses this line moving from the first sheet to the second, W will move continuously from the first W-sheet to the second—and conversely—the two sheets are connected by the single branch-line $(1, -2\pi)$ to $(+\infty, -2\pi)$.

In the same way a third τ-sheet can be constructed corresponding to the third W-sheet which can be superposed upon the second and connected with it along the line proceeding from the branch-point $(1, -4\pi)$; and so on, indefinitely. The τ-surface has infinitely many sheets, each sheet has two branch-points, $(1, -2n\pi)$ and $(1, -2(n+1)\pi)$, the first connecting it with the sheet below and the second connecting it with the sheet above. If W describes a circle about the origin passing from sheet to sheet so that φ always increases, the trochoid described by τ will pass upwards through the sheets of the τ-surface. There is a one-to-one correspondence between the sheets of the two surfaces, and a one-to-one correspondence between the values in the two corresponding sheets.

Real values of the functions $L(\tau)$ and its derivative $L'(\tau)$ corresponding

to real values of the independent variable τ are given in the accompanying tables for that branch of the function for which the derivative is positive.

VALUES OF $L(\tau)$.

τ	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.	1.000	1.516	1.772	1.986	2.179	2.357	2.527	2.690	2.846	2.998
2.	3.146	3.291	3.433	3.573	3.711	3.847	3.981	4.114	4.246	4.376
3.	4.505	4.633	4.761	4.886	5.011	5.136	5.261	5.384	5.506	5.628
4.	5.749	5.870	5.990	6.110	6.229	6.348	6.467	6.585	6.702	6.820
5.	6.937	7.053	7.170	7.286	7.401	7.517	7.632	7.747	7.862	7.976
6.	8.091	8.205	8.318	8.432	8.545	8.658	8.771	8.884	8.997	9.109
7.	9.221	9.334	9.446	9.557	9.669	9.780	9.892	10.003	10.114	10.225
8.	10.336	10.447	10.557	10.667	10.777	10.887	10.997	11.107	11.217	11.327
9.	11.437	11.547	11.657	11.766	11.875	11.984	12.093	12.202	12.311	12.420
10.	12.529	....	....	....	....	....	....	....	....	....

$L(100) = 104.651$, $L(1000) = 1006.915$

VALUES OF $L'(\tau)$.

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.	∞	2.9376	2.2950	2.0140	1.8482	1.7369	1.6549	1.5917	1.5417	1.5005
2.	1.4660	1.4365	1.4110	1.3887	1.3689	1.3512	1.3355	1.3211	1.3081	1.2962
3.	1.2853	1.2753	1.2659	1.2573	1.2493	1.2418	1.2347	1.2281	1.2219	1.2160
4.	1.2106	1.2053	1.2004	1.1957	1.1912	1.1870	1.1829	1.1790	1.1754	1.1718
5.	1.1684	1.1652	1.1621	1.1591	1.1562	1.1535	1.1508	1.1482	1.1457	1.1434
6.	1.1410	1.1388	1.1366	1.1346	1.1325	1.1306	1.1287	1.1268	1.1250	1.1233
7.	1.1216	1.1200	1.1184	1.1169	1.1154	1.1139	1.1125	1.1111	1.1097	1.1084
8.	1.1071	1.1059	1.1046	1.1034	1.1023	1.1011	1.1000	1.0989	1.0978	1.0968
9.	1.0958	1.0948	1.0938	1.0929	1.0919	1.0910	1.0902	1.0893	1.0884	1.0876
10.	1.0867	....	....	....	....	....	....	....	....	....

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SOLUTION OF THE MATRIX EQUATION $X^{-1}AX = N$.

By H. T. Burgess.

If A is any square matrix, it is a well-known theorem in the matrix theory that there exists a similar matrix N which is of a unique canonical form.* The form of N is determined by the elementary divisors of the characteristic matrix $\lambda I + A$. The existence of the matrix X is known, but no simple method of computing it is to be found in the literature. It is the object of this paper to give a simple and direct method.

In the *Annals of Mathematics*, Vol. XVIII, No. 1, September, 1916, p. 4, is found a brief paper to which reference will be made as the former paper.

1. **Notation.** Consider the linear substitution,

$$\begin{aligned} y_1 &= b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n, \\ y_2 &= b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n, \end{aligned}$$

and observe that it may be written in matrix notation in the form

$$\begin{vmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{vmatrix},$$

in which the ordinary rule of matrix multiplication applies. More briefly we may use $y = Bx$ in which small letters denote matrices of n rows and one column, and the large letter denotes a square matrix of order n. If the matrix y is zero, we have a system of n linear homogeneous equations in n unknowns expressed in this notation by $Bx = 0$.

It immediately appears from this notation that the system of linear homogeneous equations $DBx = 0$ has each and every solution of $Bx = 0$ as a solution. Further if the rank of the matrix DB is less than that of the matrix B, the system $DBx = 0$ has solutions which are not solutions of the system $Bx = 0$. If κ is any positive integer, every solution of $Bx = 0$ is also a solution of $B^kx = 0$, while any solution of $B^kx = 0$ which

* For a demonstration of this theorem, see Hilton, Linear Substitutions, p. 29.

is not a solution of $B^{k-1}x = 0$ is not a solution of $B^{\sigma}x = 0$ for $\sigma < \kappa$. This of course applies only when B is a singular matrix.

2. **Method of computation.** By Theorem 2 in the former paper, we can compute the elementary divisors of the characteristic matrix $\lambda I + A$, and thus uniquely determine the form of the canonical matrix N for any given matrix A. The nature of the canonical form N is found in the "fundamental theorem" in the former paper and need not be restated here.

From (a) $X^{-1}AX = N$, we have (b) $AX = XN$. Any solution X of (a) is also a solution of (b), and any non-singular solution of (b) is also a solution of (a). If we can devise a scheme of finding all of the solutions of (b), we have only to select a non-singular solution of (b) to obtain a solution of (a).

The necessary and sufficient conditions for X to be a solution of (b) are found by equating the corresponding elements by columns in the products AX and XN. Referring to the "fundamental theorem" in the former paper, we see that it is sufficient to consider only the t_ϵ columns corresponding to the typical component N_{t_ϵ} in N. For this purpose, we write (b) in the form $(\alpha_\epsilon I + A)X = X(\alpha_\epsilon I + N)$. It is further obvious that the conditions for these t_ϵ columns can be found by considering only the e_π columns corresponding to the typical component E_{e_π} in N_{t_ϵ}. Using the notation previously explained, we denote the block of columns in X which corresponds to these e_π columns by

$$x_i^{(\pi)} = \begin{vmatrix} x_{1i}^{(\pi)} \\ x_{2i}^{(\pi)} \\ \cdots \\ x_{ni}^{(\pi)} \end{vmatrix} \qquad (i = 1, 2, \cdots, e_\pi).$$

Since in any column of the product $(\alpha_\epsilon I + A)X$, only elements from the corresponding column of X appear, we get $(\alpha_\epsilon I + A)x_{i+1}^{(\pi)}$ in the column corresponding to the $(i+1)$st column of the component E_{e_π}. Likewise for $i > 0$, in the corresponding column of the product $X(\alpha_\epsilon I + N)$, we get simply $x_i^{(\pi)}$; for the only non-zero element in the $(i+1)$st column of $\alpha_\epsilon I + E_{e_\pi}$ is a 1 appearing in the over-principal diagonal. For $i = 0$, we get $x_0^{(\pi)} = 0$, for the first column of $\alpha_\epsilon I + E_{e_\pi}$ contains only zeros. We can now express the τ sets of equations to determine the τ blocks of columns in X which correspond to the t_ϵ columns in the component N_{t_ϵ} as follows:

$$(c) \qquad \begin{aligned} &x_i^{(\pi)} = (\alpha_\epsilon I + A)x_{i+1}^{(\pi)}, && i = 0, 1, 2, \cdots e_\pi, \\ &x_i^{(\pi)} = 0, \quad i = 0; \quad x_i^{(\pi)} \neq 0, \quad i \neq 0, && \pi = 1, 2, \cdots \tau. \end{aligned}$$

For convenience denote $\alpha_\epsilon I + A$ by B, and consider the following set of matrix equations:

$$
\begin{aligned}
x_0^{(\pi)} = Bx_1^{(\pi)} = B^2x_2^{(\pi)} = B^3x_3^{(\pi)} = \cdots = B^{\kappa-1}x_{\kappa-1}^{(\pi)} = B^\kappa x_\kappa^{(\pi)} \\
x_1^{(\pi)} = Bx_2^{(\pi)} = B^2x_3^{(\pi)} = \cdots = B^{\kappa-2}x_{\kappa-1}^{(\pi)} = B^{\kappa-1}x_\kappa^{(\pi)} \\
x_2^{(\pi)} = Bx_3^{(\pi)} = \cdots = B^{\kappa-3}x_{\kappa-1}^{(\pi)} = B^{\kappa-2}x_\kappa^{(\pi)} \\
x_3^{(\pi)} = \cdots = B^{\kappa-4}x_{\kappa-1}^{(\pi)} = B^{\kappa-3}x_\kappa^{(\pi)} \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
x_{\kappa-1}^{(\pi)} = Bx_\kappa^{(\pi)}
\end{aligned}
\tag{d}
$$

From this notation, it is obvious that this set of matrix equations is derivable from those equations which constitute the diagonal by direct matrix multiplication. If we choose $x_\kappa^{(\pi)}$ as a solution of $B^\kappa x = 0$ and not a solution of $B^{\kappa-1}x = 0$, we have $x_0^{(\pi)} = 0$, and $x_1^{(\pi)}$, $x_2^{(\pi)}$, $\cdots$ $x_\kappa^{(\pi)} \neq 0$. To obtain a solution of the sets of equations (c), we take simultaneously

$$\pi = 1, \quad 2, \quad \cdots \quad \tau,$$

$$\kappa = e_1, \quad e_2, \quad \cdots \quad e_\tau.$$

This choice is evidently always possible, for the rank of B^κ is less than the rank of $B^{\kappa-1}$ for $\kappa \leqq e_1$ by Theorem 2 of the former paper. Moreover the choice of $x_\kappa^{(\pi)}$ is by no means unique, there being an infinity of ways of making the selection.

Applying this same process to the remaining roots of the characteristic equation of A, we obtain the complete matrix X which is a solution of equation (b).

Since all the solutions of (b) are to be obtained in this manner, to obtain a non-singular solution, it is only necessary to properly choose the matrices whose selection is by no means unique.

In the application of this method, we can greatly simplify the actual computation of the elementary divisors and the matrix X if we observe first, that the reduction of any matrix B by elementary transformations on its rows only is equivalent to a matrix multiplication of the type TB where T is non-singular; secondly, that B and TB have the same rank; and thirdly, that the two systems of linear homogeneous equations $Bx = 0$ and $TBx = 0$ are equivalent. Consider the matrix equations

(1) $T_1B = R_1$ where r_1 is the rank of B and R_1,

(2) $T_2T_1B^2 = T_2R_1B = R_2$ where r_2 is the rank of B^2 and R_2,

. . . . . .

(κ) $T_\kappa \cdots T_2T_1B^\kappa = T_\kappa R_{\kappa-1}B = R_\kappa$ where r_κ is the rank of B^κ and R_κ,

. . . . . .

Let T_κ be a non-singular matrix such that $T_\kappa R_{\kappa-1}B = R_\kappa$ where R_κ is reduced by elementary transformations on its rows only, then the ranks of B^κ and R_κ are the same and the two systems of linear homogeneous equations $B^\kappa x = 0$ and $R_\kappa x = 0$ are equivalent. Furthermore by the former paper, Theorem 2, we need to carry the work out only until we find $n - r_\kappa = t_\epsilon$, where t_ϵ is the multiplicity of the factor $\lambda - \alpha_\epsilon$ in the characteristic equation of the matrix A.

3. **Illustration.** Given the matrix

$$A = \begin{vmatrix} -1 & 6 & 1 & 2 & 4 & 2 & 0 & 2 & 4 & 0 \\ 1 & 2 & 0 & 14 & 6 & 6 & -6 & 0 & 8 & -2 \\ 8 & 14 & -1 & 4 & 6 & 4 & 2 & 8 & 0 & 2 \\ 4 & 7 & 3 & -8 & 3 & -2 & 5 & 4 & -2 & 3 \\ -3 & -9 & -3 & -8 & -11 & -2 & 2 & -2 & -3 & -1 \\ 6 & 7 & 1 & -2 & 1 & 2 & 4 & 6 & 3 & 1 \\ -8 & -8 & 0 & 0 & 0 & -8 & -6 & -8 & -8 & 0 \\ 6 & 2 & 3 & -24 & -6 & -8 & 14 & 4 & -16 & 6 \\ -2 & -3 & -1 & 2 & -1 & 0 & -2 & -2 & -1 & -1 \\ 0 & -6 & -1 & -4 & -6 & 4 & 2 & 0 & 16 & -4 \end{vmatrix}.$$

Its characteristic equation is found to be $\varphi(\lambda) = (\lambda - 4)(\lambda - 3)^2(\lambda - 2)^7 = 0$, and this shows that N is of the form

$$N = \begin{vmatrix} N_1 & & & \\ & N_2 & & \\ & & & N_7 \end{vmatrix}.$$

For the first column of X, we have to solve the matrix equation

$$(4I + A)x = 0.$$

The solution found in the usual way is $x_1 = (0, 1, 0, 0, -1, 0, 0, -1, 0, 0)$.

For the second and third columns of X, we must compute, from $B = 3I + A$,

$$R_1 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \qquad r_1 = 9;$$

$$R_2 = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}, \qquad r_2 = 8.$$

By Theorem 2 of the former paper, we have

$$N_2 = \begin{vmatrix} -3 & 1 \\ 0 & -3 \end{vmatrix},$$

and our equations to determine x_2 and x_3 are

$$0 = (3I + A)x_2 = (3I + A)^2 x_3, \tag{1}$$

$$x_2 = (3I + A)x_3. \tag{2}$$

We take x_3 as a solution of $R_2x = 0$ and not a solution of $R_1x = 0$. Such a solution is obviously $x_3 = (0, 1, 1, 0, -1, 0, 0, -1, 0, -1)$, and from (2) we compute $x_2 = (1, 1, 0, 0, -1, 0, 0, -2, 0, 0)$.

For the last seven columns of X, we compute from $B = 2I + A$

$$R_1 = \begin{Vmatrix} 10 & 0 & 0 & 0 & 0 & -22 & 1 & 1 & 0 & 12 \\ 0 & 10 & 0 & 0 & 0 & 32 & 4 & 9 & 0 & -12 \\ 0 & 0 & 10 & 0 & 0 & 64 & 8 & 8 & 0 & -24 \\ 0 & 0 & 0 & 4 & 0 & 10 & -1 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 10 & -66 & -7 & -12 & 0 & 26 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix}, \qquad r_1 = 6;$$

$$R_2 = \begin{Vmatrix} 81 & 0 & 0 & 0 & -47 & 132 & 41 & 64.5 & 262 & -25 \\ 0 & 972 & 0 & 0 & 584 & -744 & -20 & 174 & -2380 & 352 \\ 0 & 0 & 486 & 0 & 584 & -744 & -20 & -312 & -2380 & 352 \\ 0 & 0 & 0 & -972 & -410 & 276 & 530 & 249 & 2320 & -94 \end{Vmatrix},$$

$$r_2 = 4;$$

$$R_3 = \begin{Vmatrix} 360 & 0 & -233 & 40 & -472 & 932 & 170 & 426 & 2210 & -276 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5177 & -3160 & 4888 & -7028 & 1510 & -2514 & -17810 & 3444 \end{Vmatrix},$$

$$r_2 = 3.$$

We have, from these ranks,

$$N_7 = \begin{Vmatrix} E_3 & & & & \\ & E_2 & & & \\ & & E_1 & & \\ & & & E_1 & \end{Vmatrix},$$

and the following equations to determine the last seven columns of X:

(1) $$0 = (2I + A)x_4 = (2I + A)^2x_5 = (2I + A)^3x_6,$$

(2) $$x_4 = (2I + A)\,x_5 = (2I + A)^2x_6,$$

(3) $$x_5 = (2I + A)\,x_6,$$

(4) $$0 = (2I + A)x_7 = (2I + A)^2x_8,$$

(5) $$x_7 = (2I + A)\,x_8,$$

(6) $$0 = (2I + A)x_9,$$

(7) $$0 = (2I + A)x_{10}.$$

We must choose x_6 as a solution of $R_3x = 0$ and not a solution of $R_2x = 0$. Such a solution is $x_6 = (390, 492, 102, 0, -492, 1, 0, -882,$

$-1, -102$). Equations (3) and (2) give

$$x_5 = (-290, -392, -98, 0, 391, 1, 0, 686, -1, 90),$$

$$x_4 = (194, 306, 112, -11, -288, 16, -32, -500, 0, -88).$$

Next we choose x_8 as a solution of $R_2x = 0$ and not a solution of $R_1x = 0$. Such a solution is

$$x_8 = (-34, -45, -10, 0, 45, 0, -1, 80, 0, 10),$$

and equation (5) gives

$$x_7 = (26, 42, 16, -1, -40, 2, -4, -68, 0, -12).$$

It remains to choose x_9 and x_{10}, as solutions of $R_1x = 0$, care being taken that they are independent of x_4 and x_7. They may be taken as

$$x_9 = (-2, -8, -16, 5, 14, 0, 20, 0, 0, 0),$$
$$x_{10} = (2, 18, 16, 5, -24, 0, 0, -20, 0, 0).$$

This gives

$$X = \begin{matrix} 0 & 1 & 0 & 194 & -290 & 390 & 26 & -34 & -2 & 2 \\ 1 & 1 & 1 & 306 & -392 & 492 & 42 & -45 & -8 & 18 \\ 0 & 0 & 1 & 112 & -98 & 102 & 16 & -10 & -16 & 16 \\ 0 & 0 & 0 & -11 & 0 & 0 & -1 & 0 & 5 & 5 \\ -1 & -1 & -1 & -288 & 391 & -492 & -40 & 45 & 14 & -24 \\ 0 & 0 & 0 & 16 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -32 & 0 & 0 & -4 & -1 & 20 & 0 \\ -1 & -2 & -1 & -500 & 686 & -882 & -68 & 80 & 0 & -20 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -88 & 90 & -102 & -12 & 10 & 0 & 0 \end{matrix}$$

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LINEAR FUNCTIONALS OF N-SPREADS.*

By Charles Albert Fischer.

It has been proved by Riesz that a linear functional of a curve, which is continuous with zeroth order, can be expressed as a Stieltjes integral.† The multiple Stieltjes integral has been defined by Fréchet and proved to be a linear functional.‡ In the first two sections of the present paper some properties of functions with finite variation, and the Stieltjes integrals determined by them, are discussed. In the last section Riesz' work is extended to functionals of n-spreads, and it is proved that when the functional is put in the form

$$A[f(x_1, \cdots, x_n)] = \int_r f(x_1, \cdots, x_n) d_n\alpha(x_1, \cdots, x_n),$$

the function $\alpha(x_1, \cdots, x_n)$ is unique if it is required to be regular. This makes it possible to extend Fréchet's work on bilinear functionals§ to functionals of n-spreads.

§ 1. **On functions with finite variation.** The variation $V_n\alpha(x)$ of a function $\alpha(x_1, \cdots, x_n)$ in the region

$$T: \qquad a_r \leqq x_r \leqq b_r \qquad (r = 1, 2, \cdots, n),$$

has been defined as the least upper bound of the expression‖

$$\sum_{i_1, \cdots, i_n} \left| \overset{(n)}{\underset{i_1, \cdots, i_n}{\Delta}} \alpha(x_1, \cdots, x_n) \right|,$$

where

$$\overset{(n)}{\underset{i_1, \cdots, i_n}{\Delta}} \alpha(x_1, \cdots, x_n) = \overset{(n-1)}{\underset{i_1, \cdots, i_{n-1}}{\Delta}} [\alpha(x_1, \cdots, x_{n-1}, x^{(i_n+1)}) - \alpha(x_1, \cdots, x^{(i_n)})],$$

and

$$\overset{(1)}{\underset{i_1}{\Delta}} \alpha(x_1, \cdots, x_n) = \alpha(x^{(i_1+1)}, x_2, \cdots, x_n) - \alpha(x^{(i_1)}, \cdots, x_n).$$

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* Read before the American Mathematical Society, December 27, 1916.

† Riesz, Annales Scientifiques de l'École Normale Supérieure, Ser. 3, Tome 31 (1914), pp. 9–14.

‡ Fréchet, Nouvelles Annales de Mathématiques, Ser. 4, Tome 10 (1910), pp. 247 and 254.

§ Fréchet, Transactions of the American Mathematical Society, vol. 16 (1915), pp. 215–34.

‖ Fréchet, loc. cit. (Nouvelles Annales), p. 242.

The multiple Stieltjes integral is defined by the equation*

$$\int_T f(x)d_n\alpha(x) = \lim_{\max(x^{(i_r+1)}-x^{(i_r)})=0} \sum_i f(\xi^{(i)}) \overset{(n)}{\Delta}_i \alpha(x), \tag{1}$$

where the arguments x, i and $\xi^{(i)}$ represent $x_1, \cdots, x_n$, $i_1, \cdots, i_n$ and $\xi^{(i_1)}, \cdots, \xi^{(i_n)}$ respectively, and $(\xi^{(i_1)}, \cdots, \xi^{(i_n)})$ is a point in the region $x^{(i_r)} \leqq x_r \leqq x^{(i_r+1)}$. If $V_n\alpha(x)$ is finite the integral exists for every continuous function $f(x)$. The function $\alpha(x)$ can always be replaced by a function which satisfies the equations

$$\alpha(x)|_{x_r=a_r} = 0, \qquad (r = 1, 2, \cdots, n),$$

without affecting its variation or the integral (1).† It will be assumed for convenience that this is always done. It follows that when r of the variables are given arbitrary values, the variation of $\alpha(x)$ considered as a function of the other $n - r$ variables cannot be greater than $V_n\alpha(x)$, and similarly $|\alpha(x)| \leqq V_n\alpha(x)$.

An $(n-1)$-flat $x_s = \xi_s$ will be called singular when the variation of $\alpha(x)$ in the region $(\xi_s - \delta \leqq x_s \leqq \xi_s + \delta;\ a_r \leqq x_r \leqq b_r,\ r \neq s)$ does not approach zero with δ.‡ It will now be proved that a necessary and sufficient condition that $x_s = \xi_s$ be singular is that $\alpha(x)$ be discontinuous in x_s at some point of T where $x_s = \xi_s$. If it is singular the variation of $\alpha(x)$ in one of the regions $(\xi_s - \delta \leqq x_s \leqq \xi_s)$ or $(\xi_s \leqq x_s \leqq \xi_s + \delta)$ must be greater than some $k > 0$ which is independent of δ. For convenience this will be assumed for the last of these regions. The whole of T can be subdivided in such a way that the inequality

$$V_n\alpha(x) - \sum_i \left| \overset{(n)}{\Delta}_i \alpha(x) \right| < \frac{k}{2} \tag{2}$$

is satisfied. It will not affect this inequality to assume that some $x^{(i_s)} = \xi_s$. Then δ can be taken as an arbitrary constant less than $x^{(i_s+1)} - \xi_s$, and the number of subdivisions increased by designating $\xi_s + \delta$ as $x^{(i_s+1)}$ and $x^{(i_s+1)}$ as $x^{(i_s+2)}$, etc. Since the inequality

$$V_n\alpha(x) - \sum_i \left| \overset{(n)}{\Delta}_i \alpha(x) \right| \geqq 0$$

is satisfied for the part of T where the inequality $\xi_s < x_s < \xi_s + \delta$ is not satisfied, the inequality (2) must be satisfied for the part of T where

* Fréchet, loc. cit. (Nouvelles Annales), p. 247.

† Fréchet, loc. cit., p. 245.

‡ It is not necessary here to distinguish between Lebesgue's singular segments, and segments containing his singular points. Lebesgue, Annales Scientifiques de l'École Normale Supérieure, Ser. 3, Tome 27 (1910), p. 410.

$\xi_s \leqq x_s \leqq \xi_s + \delta$. The inequality

$$\sum_{i_1, \cdots, i_{s-1}, i_{s+1}, \cdots, i_n} \left| \overset{(n)}{\underset{i}{\Delta}} \alpha(x) \right| > \frac{k}{2} \qquad (x^{(i_s)} = \xi_s)$$

follows. If N is the number of terms in this summation, there must be at least one of them which is greater than $k/2N$, and consequently there is a set of values for $x_1, \cdots, x_{s-1}, x_{s+1}, \cdots, x_n$ such that

$$\left| \alpha(x)|_{x_s = \xi_s + \delta} - \alpha(x)|_{x_s = \xi_s} \right| > \frac{k}{2^n N}. \tag{3}$$

Since the right member of this inequality is independent of δ, $\alpha(x)$ is discontinuous in x_s at this point. To prove the condition to be sufficient it will be assumed that an inequality such as (3) is satisfied. The left member is seen to be equal to the expression

$$\overset{(n)}{\underset{i}{\Delta}} \alpha(x) \quad (x^{(i_r)} = a_r,\ x^{(i_r+1)} = x_r,\ r \neq s;\ x^{(i_s)} = \xi_s,\ x^{(i_s+1)} = \xi_s + \delta),$$

and consequently the variation of $\alpha(x)$ in the region $(\xi_s \leqq x_s \leqq \xi_s + \delta)$ cannot approach zero with δ.

Since $V_n\alpha(x)$ is assumed to be finite the singular $(n-1)$-flats must be denumerable. If two functions $\alpha(x)$ and $\beta(x)$ are equal excepting on a denumerable set of $(n-1)$-flats $x_s = \xi_s^{(j)}$ $(j = 1, 2, \cdots)$, none of which are on the boundary of T, the equation

$$\int_T f(x) d_n\alpha(x) = \int_T f(x) d_n\beta(x)$$

must be satisfied, since T can be divided into as small subdivisions as is desired without making any $x^{(i_s)} = \xi_s^{(j)}$. It follows that the integral (1) is independent of the values assigned to $\alpha(x)$ at points where it is discontinuous in any argument. The function $\alpha(x)$ may then be assumed to satisfy the equations

$$\alpha(x) = \lim_{\epsilon = 0} \alpha(x_1, \cdots, x_r + \epsilon, \cdots, x_n) \quad (r = 1, 2, \cdots, n), \tag{4}$$

excepting on the boundary of T. Such a function, which also vanishes at $x_r = a_r$ and has finite variation, will be called regular in this paper. This is more convenient for our purposes than to make a regular function satisfy the equation $2\alpha(x) = \alpha(x + 0) + \alpha(x - 0)$, as Fréchet has done.*

§ 2. **Some properties of Stieltjes integrals.** It follows directly from the definition that the inequality

$$\left| \int_T f(x) d_n\alpha(x) \right| \leqq mf V_n\alpha(x)$$

* Fréchet, loc. cit. (Transactions), p. 217.

is always satisfied, where mf is the maximum of $|f(x)|$. It will now be proved that if $\alpha(x)$ is regular, when an $\epsilon > 0$ is given, a continuous $f(x)$ can be found which satisfies the inequality

$$\int_T f(x)d_n\alpha(x) > mf(V_n\alpha(x) - \epsilon). \tag{5}$$

The region T can be subdivided in such a way that the inequality

$$V_n\alpha(x) - \sum_i \left|\overset{(n)}{\Delta}_i \alpha(x)\right| < \frac{\epsilon}{2} \tag{6}$$

is satisfied without using any point where $\alpha(x)$ is discontinuous as an $x^{(i_s)}$, excepting perhaps on the boundary of T. Then a δ can be chosen so small that the sum of the variations of $\alpha(x)$ in the regions

$$x^{(i_s)} - \delta \leqq x_s \leqq x^{(i_s)} + \delta; \quad a_r \leqq x_r \leqq b_r, \quad r \neq s \quad (s = 1, 2, \cdots, n), \tag{7}$$

shall be less than $\epsilon/4$. The function $f(x)$ will be defined as ± 1 in each of the regions

$$x^{(i_r)} + \delta_a \leqq x_r \leqq x^{(i_r+1)} - \delta_b \qquad (r = 1, 2, \cdots, n);$$

according as $\overset{(n)}{\Delta}_i \alpha(x)$ is positive or negative, where $\delta_a = \delta$ excepting when $x^{(i_r)} = a_r$ when it is zero, and $\delta_b = \delta$ excepting when $x^{(i_r+1)} = b_r$ when it is zero. In the remainder of T, that is in the regions (7), $f(x)$ will be defined so as to be continuous everywhere and not greater than 1 in absolute value. Under these circumstances the inequality

$$\sum_i \left|\overset{(n)}{\Delta}_i \alpha(x)\right| - \int_T f(x)d_n\alpha(x) < \frac{\epsilon}{2} \tag{8}$$

must be satisfied. Inequality (5) follows directly from inequalities (6) and (8). This completes the proof that $V_n\alpha(x)$ is the least upper bound of the expression

$$\frac{1}{mf}\int_T f(x)d_n\alpha(x) \ ,$$

when $\alpha(x)$ is regular.

It follows from this theorem that if $\alpha(x)$ and $\beta(x)$ are regular and satisfy the equation

$$\int_T f(x)d_n\alpha(x) = \int_T f(x)d_n\beta(x),$$

identically in $f(x)$, they must be identical, since this equation can be put in the form

$$\int_T f(x)d_n[\alpha(x) - \beta(x)] = 0,$$

and consequently the variation of $\alpha(x) - \beta(x)$ is zero. As $\alpha(x) - \beta(x) = 0$ at $x_r = a_r$ its numerical value cannot exceed its variation.

§ 3. **Linear functionals of n-spreads.** The proof by Riesz that a linear functional of a curve can be expressed as a Stieltjes integral, which has already been referred to, can easily be extended to functionals of n-spreads. It will now be given for the case $n = 2$, and then the results stated for the more general case.

The functional $A[f(x, y)]$ will be assumed to be continuous with zeroth order,* and to be determined by the values taken by $f(x, y)$ in any region contained in T. Then $f(x, y)$ can be defined arbitrarily in the rest of T with the one condition that it be continuous. Since $A[f(x, y)]$ is linear there must be a constant M_A, independent of $f(x, y)$, such that the inequality

$$|A[f(x, y)] \leqq M_A mf$$

is satisfied. It will be assumed that M_A is the smallest such constant. As Riesz has proved for the simpler case, the field of functions for which $A[f(x, y)]$ is defined can be extended to include every $\varphi(x, y)$ which is the limit of a sequence of continuous functions which satisfy the inequalities

$$f_1(x, y) \geqq f_2(x, y) \geqq f_3(x, y) \cdots,$$

and every linear combination of such functions.† In the same way it can be proved that if there is a sequence of functions such as $\varphi(x, y)$, not necessarily continuous, which approach a limit $\psi(x, y)$ and satisfy the inequalities

$$\varphi_1(x, y) \geqq \varphi_2(x, y) \geqq \varphi_3(x, y) \cdots, \tag{9}$$

and continuous functions $f_i(x, y)$ can be found such that $f_i \geqq \varphi_i \geqq f_{i+1}$, then the equation

$$A[\psi(x, y)] = \lim_{m=\infty} A[\varphi_m(x, y)]$$

is satisfied.

A function $f(x, y; \xi, \eta)$ will be defined as identically zero when $\xi = a_1$ or $\eta = a_2$, and for other values of ξ and η as equal to 1 in the region $(a_1 \leqq x \leqq \xi; a_2 \leqq y \leqq \eta)$ and equal to 0 elsewhere. It is easily proved to be the limit of a decreasing sequence of continuous functions.‡ The function $\alpha(x, y)$ can then be defined by the equation

$$\alpha(\xi, \eta) = A[f(x, y; \xi, \eta)].$$

* If the continuity is of a higher order see Fischer, Bulletin of the American Mathematical Society, vol. 23 (1916), p. 88.

† Riesz, loc. cit., pp. 10–13.

‡ Compare with Riesz, loc. cit., p. 13.

It follows from the definition of $f(x, y; \xi, \eta)$ that $\alpha(a_1, y) = \alpha(x, a_2) = 0$. To prove that it is regular, a sequence of decreasing constants $\epsilon_1, \epsilon_2, \cdots$ will be chosen approaching zero as a limit. Then the sequences $A[f(x, y; \xi + \epsilon_m, \eta)]$ and $A[f(x, y; \xi, \eta + \epsilon_m)]$ both approach $\alpha(\xi, \eta)$, and the inequalities analogous to (9) are satisfied. It follows directly that the equations analogous to (4) are satisfied and $\alpha(x, y)$ is regular. By definition $\underset{i,j}{\Delta} f(x, y; \xi, \eta) = 1$ in the region

$$T_{ij}: \quad \xi_i < x \leqq \xi_{i+1}; \qquad \eta_j < y \leqq \eta_{j+1}, \tag{10}$$

and vanishes everywhere else. If $i = 0$ the first of these inequalities should read $a_1 \leqq x \leqq \xi_1$, and similarly if $j = 0$. The inequality

$$\sum_{i,j} \left| \underset{ij}{\Delta} \alpha(\xi, \eta) \right| = A[\sum_{i,j} \pm \underset{i,j}{\Delta} f(x, y; \xi, \eta)] \leqq M_A$$

follows. Thus the variation of $\alpha(x, y)$ cannot be greater than M_A.

It will now be proved that the equation

$$A[f(x, y)] = \int_T f(x, y) d_2 \alpha(x, y) \tag{11}$$

is satisfied. The region T will be divided arbitrarily into the set of rectangles T_{ij}, and an arbitrary point of T_{ij} designated by (ξ_i', η_j'). A function $\varphi(x, y)$ will then be defined by the equation

$$\varphi(x, y) = \sum_{i,j} f(\xi_i', \eta_j') \underset{i,j}{\Delta} f(x, y; \xi, \eta).$$

It follows that

$$A[\varphi(x, y)] = \sum_{i,j} f(\xi_i', \eta_j') \underset{i,j}{\Delta} \alpha(\xi, \eta).$$

The definition of $\varphi(x, y)$ implies that it is equal to $f(\xi_i', \eta_j')$ in T_{ij}. Then if ω is the greatest oscillation of $f(x, y)$ in any T_{ij}, the inequality

$$\left| A[f(x, y)] - \sum_{i,j} f(\xi_i', \eta_j') \underset{i,j}{\Delta} \alpha(\xi, \eta) \right| = |A[f - \varphi]| \leqq \omega M_A$$

must be satisfied, and equation (11) follows.

The proof for the general case is essentially the same as the above and need not be repeated here. The results of the last two sections may be stated in the following way. *If a linear functional $A[f(x)]$, which is continuous with zeroth order, is determined by the values taken by $f(x_1, \cdots, x_n)$ in the region T, then there is a unique regular function $\alpha(x_1, \cdots, x_n)$, independent of f, which satisfies the equation*

$$A[f(x)] = \int_T f(x) d_n \alpha(x),$$

and the variation of α is the least upper bound of the absolute value of $A[f(x)]/mf$.

As this same property of functionals of curves is the foundation of Fréchet's work on bilinear functionals,* his proof that a bilinear functional can be expressed as a double Stieltjes integral can also be extended to functionals of n-spreads.

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* Fréchet, loc. cit. (Transactions).

CORRECTION.

BY L. I. HEWES.

On p. 197, Vol. XVIII, the equation in Fig. 1 should be

$$Z_1Z_2 - Z_3 + \sqrt{1 + Z_2^2}\sqrt{1 + Z_1^2} = 0$$

and Z_1 in the curve net should read Z_3.

SUBSTITUTION GROUPS AND POSSIBLE ARRANGEMENTS OF THE PLAYERS AT CARD TOURNAMENTS.

By G. A. Miller.

1. Introduction. The general problem of finding $4n - 1$ successive arrangements of $4n$ players at n tables so that each player shall have each of the others once as a partner and twice as an opponent during a series of $4n - 1$ games was treated by E. H. Moore in his "Tactical Memoranda III," where several general theorems were established and references to the literature on the subject were given.* The chief aims of the present article are to explain a different method of solving this problem when n is a power of 2, and to emphasize the elements which it has in common with the theory of substitution groups.

While the main elements of novelty of the present paper relate to methods, it may be desirable to direct attention to an interesting result proved towards the end of the following section, which does not seem to have been noted before. This result may be stated as follows: When n is an even power of 2 it is always possible to find two quite distinct sets of $4n - 1$ arrangements. In one of these sets all the players at every table are the same in two arrangements whenever two of the players at some one of the tables are the same, while in the other set the four players at a table are never the same during two of the $4n - 1$ games.

It is evident that 8 of the possible 24 permutations of the players at a table will result in arrangements in which the players have the same partners. These eight permutations constitute the octic group and the corresponding arrangements of the players will not be regarded as distinct in what follows unless the contrary is explicitly stated. Hence the 24 possible permutations of the players at a table give rise to only three distinct arrangements of these players according to our ordinary use of this term. It is clear that in these three arrangements each player has each of the other three once as a partner and twice as an opponent.

2. Number of the players of the form 2^m. When the number of the players is of the form 2^m, $m > 1$, a possible series of $2^m - 1$ arrangements such that each player shall have each of the others once as a partner and twice as an opponent can always be obtained directly by means of the regular substitution group G which represents the abelian group of order

* E. H. Moore, American Journal of Mathematics, vol. 18 (1896), p. 290.

2^m and of type $(1, 1, 1, \cdots)$. A method of finding such a series of arrangements is as follows:

Select any one of the subgroups of order 4 contained in G and let the letters of each of its 2^{m-2} transitive constituents represent the players at one table. Arrange these players so that the transpositions of one of the substitutions of order 2 contained in the selected subgroup of order 4 represent the partners. The other two substitutions of order 2 contained in this subgroup will then represent the opponents in the initial arrangement of the 2^m players. To obtain the remaining $2^m - 2$ arrangements of the series in question it is only necessary to transform this initial arrangement successively by the powers of any substitution s of order $2^m - 1$ in the holomorph of G.

To prove that these transforms constitute $2^m - 1$ arrangements in which each player has each of the others once as a partner and twice as an opponent it is only necessary to observe that the substitutions of G involve every possible transposition on 2^m letters once and only once, and that the powers of s transform every one of the substitutions of order 2 in G into every other one once and only once. The former of these two statements is evidently true, and the latter can easily be proved by noting that s is in a co-set of the holomorph of G, with respect to the invariant subgroup G, which involves a cyclic substitution of order $2^m - 1$. In fact, a substitution of this order is known to appear in the group of isomorphisms of G and every power of it which does not reduce to the identity must involve $2^m - 1$ letters.

Since the powers of s transform the substitution whose transpositions represent partners in the initial arrangement into every other substitution of order 2 contained in G, each of the 2^m players will have each of the others once as a partner in the arrangements which are conjugate to the initial arrangement under the powers of s. Moreover, since the powers of s transform also each of the two substitutions which represent the opponents in the initial arrangement into the same set of substitutions, each player will have each of the others twice as an opponent in this set of conjugate arrangements. In fact, it is not difficult to see that he will have each of the others once as an opponent of the first kind and once as an opponent of the second kind in the sense of a "triple-whist-tournament arrangement" used by E. H. Moore in the article to which we referred, if we employ his method of assigning order to the opponents.

To illustrate the method explained above, and to exhibit its direct availability for the purpose of finding the specified arrangements of 2^m players during a series of $2^m - 1$ games, we shall consider the special case when $2^m = 8$. Writing G in the following form:*

* The Collected Mathematical Papers of Arthur Cayley, vol. 13 (1897), p. 139.

1	$ae \cdot bf \cdot cg \cdot dh$
$ab \cdot cd \cdot ef \cdot gh$	$af \cdot be \cdot ch \cdot dg$
$ac \cdot bd \cdot eg \cdot fh$	$ag \cdot bh \cdot ce \cdot df$
$ad \cdot bc \cdot eh \cdot fg$	$ah \cdot bg \cdot cf \cdot de$

and using the subgroup composed of its first four substitutions to determine the initial arrangement, the transpositions of the first substitution of order 2 representing the partners, we may denote the initial arrangement as follows:

$$\begin{array}{ccc} & c & \\ a & & b \\ & d & \end{array} \qquad \begin{array}{ccc} & g & \\ e & & f. \\ & h & \end{array}$$

It is well known that the group of isomorphisms of the special G under consideration is the simple group of order 168 and that any one of the 48 automorphisms of G in which no operator of order 2 corresponds to itself gives rise to a substitution which can be used for s. It is easy to verify that *beghdfc* is such a substitution, and by transforming the noted initial arrangement by the successive powers of this substitution we obtain the following six additional arrangements:

$$\begin{array}{cccc} & b & & h \\ a & & g & \\ & f & & d \\ & h & & c \\ a & & f & \\ & e & & b \end{array} \quad \begin{array}{cccc} & e & & d \\ a & & h & \\ & c & & f \\ & d & & b \\ a & & c & \\ & g & & e \end{array} \quad \begin{array}{cccc} & g & & f \\ a & & d & \\ & b & & c \\ & f & & e \\ a & & b & \\ & h & & g \end{array}$$

These six arrangements together with the noted initial arrangement constitute therefore *seven successive arrangements of eight players at two tables so that each one has each of the others once as a partner and twice as an opponent.* By making a list for each player, denoting his partner and table for each of the seven successive games, these arrangements can be easily followed in practice.

To obtain a set of 15 successive arrangements of 16 players so that each one has each of the others once as a partner and twice as an opponent, we may start with any subgroup of order 4 in the regular abelian substitution group of order 16 and of type (1, 1, 1, 1). This group has been studied extensively, and in a well-known notation* such a subgroup gives rise to the following initial arrangement:

$$\begin{array}{ccccccccc} & c & & g & & k & & o & \\ a & & b & & e & & f & & n. \\ & d & & h & & l & & p & \end{array}$$

* Cf. American Journal of Mathematics, vol. 20 (1898), p. 229.

By transforming this arrangement by the successive powers of the substitution $s = bfdjinohpcmlegk$ which is known to occur in the group of isomorphisms of this G* we obtain 15 conjugate arrangements which satisfy the stated conditions.

It may be observed that only 15 out of the 35 subgroups of order 4 contained in the said group of order 16 appear in the conjugates under consideration, and hence it is possible to obtain different sets of arrangements by this method when G is given and $m > 3$. In particular, if we had started with a subgroup of order 4 which is transformed into itself by s^5 the resulting 15 arrangements could have been combined into 5 sets of three corresponding to the 5 conjugate subgroups of order 4 under the powers of s. That is, these 15 arrangements are such that whenever two players appear at the same table in two different arrangements then the four players at each table are the same in these two arrangements. Such sets of arrangements are always possible when m is even since the number of subgroups of order 4 contained in G cannot then be divisible by $2^m - 1$. Hence the preceding method gives rise to at least two very different sets of arrangements whenever m is even and > 2.

3. **Number of players not a power of 2.** When the number of the players is not of the form 2^m the preceding method clearly can not be employed since it is impossible in this case to construct a group whose order is equal to the number of players and which involves only substitutions of order 2 besides the identity. For the sake of exhibiting contact with what precedes it seems, however, desirable to refer here briefly to a very elementary general method notwithstanding the facts that this method is not essentially new and does not always lead directly to arrangements which satisfy the stated conditions completely.

The method in question is based on the fact that when we adjoin to a regular cyclic substitution group C of order $4n - 1$ a substitution of order 2, which involves no additional letters and transforms each of the substitutions of C into its inverse, there results a dihedral group whose $4n - 1$ substitutions of degree $4n - 2$ and of order 2 contain each transposition on the $4n - 1$ letters involved in C once and only once. By adding to any one of these substitutions of order 2 a transposition composed of the letter of C which this substitution omits and an arbitrary letter not involved in C, there results a substitution of degree $4n$ whose $4n - 1$ conjugates under C contain each of the possible transpositions on $4n$ letters once and only once.

Hence it is evident that if the transpositions of any one of these substitutions of degree $4n$ represent the partners, arranged in the usual way around rectangular tables, the $4n - 1$ conjugates of such an arrangement

* Loc. cit.

under the substitutions of C constitute $4n-1$ successive arrangements of the $4n$ players such that each of the players has each of the others once and only once as a partner. Since the substitutions representing the opponents in the initial arrangement are not contained in the said dihedral group we cannot use the method of the preceding section to determine the different opponents of each player during the series of games under consideration. It is evident that the player represented by the letter not contained in C has each of the others twice as an opponent in these conjugates.

To determine the opponents of the other players let

$$t = a_1 a_2 \cdots a_{4n-1}$$

represent a generating substitution of C, and note the differences between the subscripts of the letters representing the opponents in the initial arrangement. A necessary and sufficient condition that a given player have the same player k times as an opponent in the noted set of conjugates is that k of these differences be congruent modulo $4n-1$. Hence it results that if we can arrange the players for the initial game so that no more than two of these differences are congruent modulo $4n-1$ each of the $4n$ players will have each of the others twice as an opponent in the series of $4n-1$ arrangements under consideration. It is evident that this condition is satisfied in the following initial arrangement for the special case when $n=7$:

$$\begin{array}{cccccccccccccc} & a_7 & & a_{14} & & a_{10} & & a_8 & & a_6 & & a_{12} & & a_{13} \\ a_1 & & a & & a_2 & a_{27} & a_3 & a_{26} & a_4 & a_{25} & a_5 & a_{24} & a_9 & a_{20} & a_{11} & a_{18} \\ & a_{22} & & a_{15} & & a_{19} & & a_{21} & & a_{23} & & a_{17} & & a_{16} \end{array}$$

Sometimes this condition cannot be directly satisfied but a few trials will lead to an initial arrangement such that each player will have almost all of the others twice as a partner in the series of $4n-1$ conjugates considered above. For a complete solution of some of these cases we refer to the article noted in the Introduction, since our main object has been to call attention to results, relating to the arrangements of players at card tournaments, which follow directly from properties of well known substitution groups. We shall merely add that when $n=3$ the conjugates of the following initial arrangement under the powers of t give a set of arrangements which is not identical with either of the two sets of arrangements for 12 players noted in the article just cited:

$$\begin{array}{ccccccc} & a_7 & & a_6 & & a_5 & \\ a_1 & & a & a_2 & a_9 & a_3 & a_4. \\ & a_{10} & & a_8 & & a_{11} & \end{array}$$

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PROBLEMS OF THE CALCULUS OF VARIATIONS INVARIANT UNDER A CONTINUOUS GROUP.

By I. A. Barnett.

In order to make clear the purpose of this paper, it will be well to recall the results of a few of the typical problems in the calculus of variations. It is known that in the problem of the minimum surface of revolution, the integral to be minimized is

$$I = \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\,dx,$$

and the extremals are of the form

$$y - \alpha = a\varphi\left(\frac{x - b}{a}\right) \qquad (a > 0), \tag{1}$$

where, in this problem, the function φ is the hyperbolic cosine and $\alpha = 0$. In case of the brachistochrone problem, the integral is

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{\sqrt{y - \alpha}}\,dx$$

and has for its extremals the cycloids

$$x = a(\theta - \sin\theta) + b, \qquad y = a(1 - \cos\theta) + \alpha,$$

which, if θ is eliminated, have the form (1). For Newton's problem of the surface of minimum resistance,

$$I = \int_{x_1}^{x_2} \frac{yy'^3}{1 + y'^2}\,dx$$

and the extremals are

$$x - b = a\left(\log p + \frac{1}{p^2} + \frac{3}{4p^4}\right), \qquad y = a\left(\frac{1}{p^3} + \frac{2}{p} + 1\right),$$

which can also be put in the form (1) by the elimination of p.

In these problems, and others, the equations of the extremals all come under the type (1) and the curves represented by (1) all have the property of being invariant under the two-parameter group of stretchings and

translations

$$x' = sx + t, \qquad y' - \alpha = s(y - \alpha) \qquad (\alpha = \text{const.}), \tag{2}$$

a fact that can be easily verified.

It is proposed in this paper, first, to find the types of integrals

$$I = \int_{x_1}^{x_2} f(x, y, y')dx, \tag{3}$$

whose extremals are invariant under the group (2).* It is then desired to specialize further these integrals by prescribing that the transversality relation† is equivalent to

$$\bar{y}' = P(y') \tag{4}$$

where $P(y')$ is a given function.‡

In the last half of the paper, there is given a geometrical construction of the focal points for the extremals (1).

§ 1. **The types of integrals whose extremals are invariant under the group (2).** In order to apply to this problem a method which Darboux§ has developed, the most general differential equation of the second order which is invariant under (2) will be required. Two independent infinitesimal transformations of the group (2) are

$$U_1 f \equiv \frac{\partial f}{\partial x} \equiv p, \qquad U_2 f \equiv x\frac{\partial f}{\partial x} + (y - \alpha)\frac{\partial f}{\partial y} \equiv xp + (y - \alpha)q, \tag{5}$$

with the twice extended infinitesimal transformations

$$U_1'' f \equiv p, \qquad U_2'' f \equiv xp + (y - \alpha)q - y''q'' \qquad \left(q'' \equiv \frac{\partial f}{\partial y''}\right). \tag{6}$$

A necessary and sufficient condition that the solutions of the equation

$$y'' = A(x, y, y')$$

* For an analogous problem, see Guldberg, "Sur une classification des problèmes du calcul des variations," Rendiconti di Circolo Matematico di Palermo, vol. 21 (1906), pp. 66–74.

† The direction $\bar{y}'$ is said to be transversal to the direction y' with respect to a function $f(x, y, y')$ if

$$f(x, y, y') + (\bar{y}' - y')f_{y'}(x, y, y') = 0.$$

‡ Stromquist, "A second inverse problem of the calculus of variations," Annals of Mathematics, Series 2, vol. 9 (1908), pp. 57–68.

§ Darboux, Leçons sur la theorie générale des surfaces, vol. 3, p. 53.

shall be invariant under $U_i''f$ $(i = 1, 2)$ is that $U_i''f = 0$ whenever $f \equiv y'' - A(x, y, y') = 0$. One finds easily from these conditions that A must have the form

$$A = \frac{F(y')}{y - \alpha}, \tag{7}$$

where F is an arbitrary function.

Hence, *the most general differential equation of the second order which is invariant under the group* (2) *has the form*

$$(y - \alpha)y'' = F(y'). \tag{8}$$

The solutions of this equation are readily found to have the form (1), which proves that these curves constitute the most general two-parameter family invariant under the group (2).

To determine the integrals whose extremals are defined by equations (1), the value y'' given by equation (8) may be substituted in Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

when differentiated out. The result is the partial differential equation for the determination of f,

$$f_{y'y'}F(y') + y'(y - \alpha)f_{y'y} + (y - \alpha)f_{y'x} - (y - \alpha)f_y = 0.$$

If Darboux's method is now applied to the solution of this equation, one obtains the following result.

The most general function f, whose extremals are invariant under the group (2), *is*

$$f = \int_0^{y'} \frac{y' - v}{F(v)} G[(y - \alpha)\rho(v),\ x - (y - \alpha)\sigma(v)]dv + \lambda_0 y' + \mu_0 + y'\frac{\partial H}{\partial y} + \frac{\partial H}{\partial x}, \tag{9}$$

where F and G are arbitrary functions of their arguments, H is an arbitrary function of x and y, ρ and σ are functions defined by

$$\rho(y') = \frac{1}{\varphi[\psi(y')]}, \qquad \sigma(y') = \frac{\psi(y')}{\varphi(y')}, \qquad \varphi_u[\psi(y')] = y',$$

and λ_0 and μ_0 are particular solutions of the equation

$$G[(y - \alpha)\rho(0),\ x - (y - \alpha)\sigma(0)] + \frac{\partial \lambda}{\partial x} - \frac{\partial \mu}{\partial y} = 0.$$

§ 2. Integrals with invariant extremals and a prescribed transversality relation. If it is required that the integrals (3) have a transversality relation of

the form (4), it is clear that f must satisfy the equation

$$f(x, y, y') + (P - y')f_{y'}(x, y, y') = 0, \tag{10}$$

which must hold for all values of x, y, y'. The solution of this linear partial differential equation has the form*

$$f = \eta(x, y)\theta(y'), \tag{11}$$

where η is an arbitrary function and

$$\log \theta(y') = \int_0^{y'} \frac{dy'}{y' - P(y')}.$$

Two methods of procedure are now possible in order to determine the function f with extremals invariant under the group (2) and having the transversality relation (4). The value of f given by the expression (9) may be compared with that defined by (11) and one may thus restrict the arbitrary functions F, G, H. A second and far simpler method, however, is to obtain Euler's equation corresponding to the function f in expression (11), and then to subject this differential equation to the invariance of the group (2). In this way conditions on the function η may be obtained.

Euler's differential equation corresponding to the function given by (11) is found to be

$$y'' = \frac{\eta_y(\theta - y'\theta') - \eta_x\theta'}{\eta\theta''} \equiv S(x, y, y'), \tag{12}$$

where $\theta' = d\theta/dy'$, $\theta'' = d^2\theta/dy'^2$. Let us suppose that θ'' is everywhere different from zero, so that the differential equation (12) is perfectly well defined. If the differential equation (12) is invariant under (2), it can be easily argued† that the partial differential equation

$$Af \equiv p + y'q + S(x, y, y')q' = 0 \qquad \left(q' = \frac{\partial f}{\partial y'}\right) \tag{13}$$

is invariant under the first extended group

$$U_1'f \equiv p,$$

$$U_2'f \equiv xp + (y - \alpha)q,$$

which, in this case, has the same infinitesimal transformations as the original group (2).

A necessary and sufficient condition that equation (13) shall be invariant under $U_i'f$ $(i = 1, 2)$ is that there shall exist functions $\mu_i(x, y, y')$ such that

$$(AU_i')f \equiv AU_i'f - U_i'Af = \mu_i(x, y, y')Af. \tag{14}$$

* Stromquist, loc. cit.

† Lie, Vorlesungen über Differential Gleichungen, p. 364.

Using the first of these relations, one finds that

$$-\frac{q'}{\eta^2\theta''}[\eta\{(\theta - y'\theta')\eta_{yx} - \eta_{xx}\theta'\} - \eta_x\{\eta_y(\theta - y'\theta') - \eta_x\theta'\}]$$
$$= \mu\left[p + qy' + \frac{\eta_y(\theta - y'\theta') - \eta_x\theta'}{\eta\theta''}q'\right]$$

which is to be an identity in p, q, q', and hence $\mu \equiv 0$. The function η must then satisfy the equation

$$(\eta\eta_{yx} - \eta_x\eta_y)(\theta - y'\theta') - (\eta\eta_{xx} - \eta_x^2)\theta' = 0. \tag{15}$$

Remembering that $\theta'' \neq 0$ and hence that $\theta' \neq$ const., and keeping in mind, moreover, that x, y and y' are independent variables in this last equation, one finds that $(\theta - y'\theta')/\theta' =$ const. different from zero, or else the two equations $\eta\eta_{xx} - \eta_x^2 = 0$ and $\eta\eta_{yx} - \eta_x\eta_y = 0$ hold simultaneously.

In the latter case, one has

$$\frac{\partial}{\partial x}\left(\frac{\eta_y}{\eta}\right) = 0, \qquad \frac{\partial}{\partial x}\left(\frac{\eta_x}{\eta}\right) = 0,$$

from which it is easily found that η must have the form

$$\eta = me^{kx}Y(y),$$

where m and k are arbitrary constants and Y is an arbitrary function of y.

The partial differential equation (13) now reduces to

$$Af \equiv p + y'q + \frac{Y_y(\theta - y'\theta') - kY\theta'}{Y\theta''}q' = 0.$$

Making use now of the second of the relations (14), one finds after a little computation that Y must satisfy the equation

$$\frac{\theta - y'\theta'}{\theta'}\left\{\frac{Y_y}{Y} + (y - \alpha)\frac{YY_{yy} - Y_y^2}{Y^2}\right\} = k.$$

Since in the case under consideration $(\theta' - y'\theta')/\theta' =$ const. has been excluded, the equations $k = 0$ and

$$\frac{d}{dy}\left[(y - \alpha)\frac{d}{dy}\log Y\right] = 0$$

must hold true, from which it follows that

$$Y = \gamma(y - \alpha)^\beta,$$

where β and γ are arbitrary constants of integration.

Hence, *all the integrals* (3), *whose extremals are invariant under the group* (2) *and which have the prescribed transversality relation* (4), *are of the form*

$$I = \gamma \int_{x_1}^{x_2} (y - \alpha)^{\beta} \theta(y') dx, \tag{16}$$

where β *and* γ *are arbitrary constants and*

$$\log \theta(y') = \int_{0}^{y'} \frac{dy'}{y' - P(y')}.$$

On recalling now that

$$\theta - y'\theta' = c\theta' \qquad (c = \text{const.})$$

also satisfies equation (15), one readily finds that

$$\theta = (y' + c)d,$$

where d is a constant. This is an expression linear in y' and hence $\theta'' \equiv 0$. It can be easily shown in this case that it is not possible to have a two-parameter family of extremals, and this is not in accordance with the original hypothesis. There still remains the possibility that θ'' may vanish at some isolated points of the extremals, but this is of rare occurrence in the ordinary problems of the calculus of variations and will not be considered here.

As an application of the foregoing results, it may be interesting to remark that when $P = -1\,y'$, that is, when the extremals and transversals are perpendicular to each other, $\theta(y')$ takes the form

$$\theta(y') = \sqrt{1 + y'^2},$$

and, hence, the integrals (16) are

$$I = \gamma \int_{x_1}^{x_2} (y - \alpha)^{\beta} \sqrt{1 + y'^2}.$$

This case includes some of the commonest problems of the calculus of variations. A few examples are the problem of finding the shortest distance from a point to a curve, the problem of finding the curve joining a point and a curve along which the integral $\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx/y$ shall have the least value, and all the problems mentioned at the beginning of this paper considered with one end point variable, except Newton's problem.

§ 3. **A geometrical construction of focal points for extremals invariant under the above group.** It is well known in the calculus of variations that if a particular extremal is imbedded in a one-parameter family of extremals

$$E_t: \quad y = e(x, t),$$

every one of which is cut transversally by a curve

$$G: \quad x = x(t), \qquad y = y(t),$$

then the focal point of G on a particular arc E_0 of the family is the point of contact of E_0 with the envelope of the family E_t. It is proposed to give, in this section, a construction of the focal points which holds for all families of extremals of the type

$$y = a\varphi\left(\frac{x-b}{a}\right) + \alpha \equiv a\varphi(u) + \alpha, \tag{17}$$

invariant under the group (2) for problems which have the transversality relation

$$\frac{y_t}{x_t} = -\frac{1}{Q(p)}, \tag{18}$$

where now p is used instead of y' to denote the slope along an extremal at the point (x, y), and Q represents the slope of the normal to the curve G at its point of intersection with the extremal. It is interesting to note in this connection that Miss M. E. Sinclair has given a geometrical construction of the focal points when the extremals are catenaries and the condition of transversality is orthogonality.*

In order to obtain a valid equation for the determination of the focal points, it will be convenient, in the first place, to exclude from this discussion all those transversal curves which have a horizontal tangent at the point P_1 where the extremal and transversal curves intersect. Secondly, Q must be a monotonic function of p in the neighborhood of this point of intersection, i. e., $Q'[\varphi_u(u)]$ must not vanish at $u = u_1$ (u_1 is the value of the variable u corresponding to P_1) nor in its neighborhood.

Consider, now, the one-parameter family of curves obtained from (17) whose individual curves intersect G and satisfy the relation (18) with G at the point of intersection. Then, after some computation, the contact points of those curves with their envelope are found to be determined by the equation†

$$\frac{\omega(u) - \Omega_1}{\omega(u_1) - \Omega_1} = \frac{r_1\varphi_u(u_1)}{r_1\varphi_u(u_1) + \epsilon S_1}, \tag{19}$$

where

$$\omega(u) = \frac{\varphi(u)}{\varphi_u(u)} - u,$$

$$\Omega_1 = -[u_1 + \varphi(u_1)Q(p_1)],$$

* Miss M. E. Sinclair, "On the minimum surface of revolution in the case of one variable end-point," *Annals of Mathematics*, series 2, vol. 8 (1907), p. 177.

† Bolza, Lectures on the Calculus of Variation, p. 113.

$$S_1 = \rho_1 Q_1' \left[\frac{1 + \varphi_u^2(u_1)}{1 + Q_1^2} \right]^{3/2}, \qquad Q_1 = Q(p_1),$$

ρ_1 is the radius of curvature of G, r_1 the radius of curvature of E at the point P_1, and $\epsilon = y_t/|y_t|$.

It is now desired to obtain a geometrical interpretation of equation (19). For the sake of concreteness, let it be supposed that the parametric representation of the curve G is of such a nature that as t increases, y decreases in the neighborhood of the point P_1. Then $y_t < 0$ and $\epsilon = -1$. Moreover, in accordance with the usual convention,* the positive directions of the tangent and normal to G at the point P_1 will be like those indicated in Fig. 1. It will be understood in the following drawings that the primed letters refer to the curve G while the unprimed letters to the extremal E.

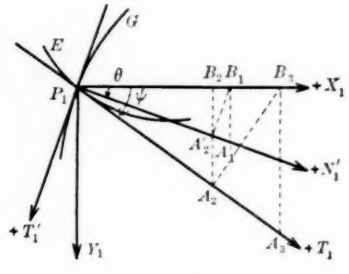

FIG. 1.

In Fig. 1

$$\theta = \tan^{-1} Q_1, \qquad \psi = \tan^{-1} \varphi_u(u_1),$$

$$P_1A_1' = \rho_1 Q_1', \qquad P_1B_2 = P_1A_1' \cos^3\theta = \frac{\rho_1 Q_1'}{(1 + Q_1^2)^{3/2}},$$

$$P_1A_3 = P_1B_2 \sec^3\psi = \rho_1 Q_1' \left[\frac{1 + \varphi_u^2(u_1)}{1 + Q_1^2} \right]^{3/2} = S_1.$$

The construction of the quantity S_1 is then as follows:

On the normal to the curve G at the point P_1, lay off $P_1A_1' = \rho_1 Q_1'$† taking care of signs. Project P_1A_1' upon P_1X_1, a line through P_1 parallel to the X-axis, and so determine B_1. Project B_1P_1 back again upon the normal and through A_2', the terminal point of the last projection, draw a perpendicular to P_1X_1. Call A_2 the intersection of this perpendicular with the tangent to the extremal at the point P_1. The intersection of a perpendicular to the tangent at A_2 with P_1X_1 will determine the point B_3, while the inter-

* See Scheffers, Anwendungen der Differential und Integralrechnung auf Geometrie, vol. 1, p. 37.

† This quantity can be constructed geometrically if the curve $Q(p)$ is supposed drawn.

section of a perpendicular to P_1X_1 at B_3 with the tangent will, in turn, give A_3. $P_1A_3 = S_1$ is the required quantity.

Having now obtained S_1, one may easily complete the construction of the focal points. It is to be understood, of course, that while the drawing in Fig. 2 is made only for a special case, the construction as given below is perfectly general.

In Fig. 2,

$$P_1C_1 = r_1, \qquad P_1R_1 = -r_1\varphi_u(u_1), \qquad P_1A_3 = \epsilon S_1,$$

$$P_1L_1 // P_1T_1', \qquad TL_1 // P_1T_1.$$

$$a[\omega(u) - \Omega_1] = \frac{y - \alpha}{\varphi_u(u)} + (x_1 - x) + (y_1 - \alpha)Q_1$$

$$= TF + FF_1 + F_1T_1' = TT_1', \tag{20}$$

$$a[\omega(u_1) - \Omega_1] = \frac{y_1 - \alpha}{\varphi_u(u_1)} + (y_1 - \alpha)Q_1 = T_1F_1 + F_1T_1' = T_1T_1'.$$

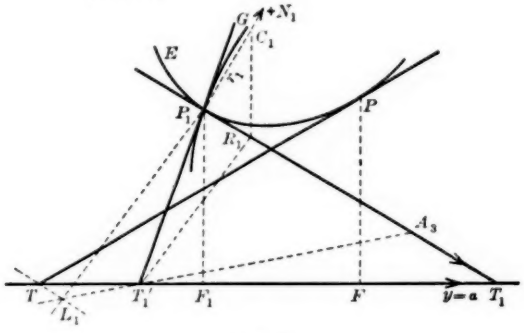

FIG. 2.

If one substitutes the expressions given by (20) in equation (19) he obtains

$$\frac{TT_1'}{T_1T_1'} = \frac{R_1P_1}{R_1P_1 + P_1A_3} = \frac{R_1P_1}{R_1A_3},$$

which, as can be seen from Fig. 2, is a true relation. Hence, the construction may be made as follows:

Draw the tangent and normal to the extremal E at the point P_1 and give these lines positive directions in accordance with the usual conventions. On the normal, lay off with proper sign $P_1C_1 = r_1$, and on the tangent make $P_1A_3 = \epsilon S_1$. Through C_1 draw a perpendicular to $y = \alpha$ and call its intersection with P_1T_1, R_1. Construct the tangent to G at the point P_1 and call T_1' its intersection with $y = \alpha$. Connect T_1' with both A_3 and R_1. Through the intersection L_1 of $T_1'A_3$ and a line P_1L_1 parallel to R_1T_1' draw TL_1 parallel to P_1T_1. The line TP tangent to E determines the focal point P.

The case $\rho_1 = 0$ possesses particular interest since then $S_1 = 0$, A_3 coincides with P_1, T coincides with T_1, and the construction reduces to the Lindelöf construction* for the point conjugate to P_1. The equation for the determination of the focal point is easily found from equation (19) to be $\omega(u) - \omega(u_1) = 0$. There are similar simplifications when the transversal curve is a straight line, i. e., when $\rho_1 = \infty$ and also when ρ_1 satisfies the equation $r_1\varphi_u(u_1) + \epsilon S_1 = 0$.

As has already been remarked, the case when the extremals and their transversals are orthogonal is of very frequent occurrence in the calculus of variations, and, hence the construction of the focal in this case deserves particular mention. Since now $Q_1 = p_1$, S_1 reduces to ρ_1, and the equation for the determination of the focal points becomes

$$\frac{\omega(u) - \Omega_1}{\omega(u_1) - \Omega_1} = \frac{r_1\varphi_u(u_1)}{r_1\varphi_u(u_1) + \epsilon\rho_1},$$

where, now

$$\Omega_1 \equiv -[u_1 + p_1\varphi(u_1)].$$

It can be easily verified that the construction (Fig. 2) is as follows:

Draw the tangent and normal to E at the point P_1. Lay off with proper sign $P_1C_1 = r_1$ on the normal and $P_1A_3 = \rho_1$ on the tangent. Construct C_1R_1 perpendicular to $y = \alpha$ and call its intersection with the tangent P_1T_1, R_1. From the point of intersection T_1' of the normal and $y = \alpha$ draw lines to A_3 and R_1. Through the intersection L_1 of $T_1'A_3$ and a line through P_1 parallel to R_1T_1' draw TL_1 parallel to P_1T_1. The line TP tangent to E determines the focal point P.

UNIVERSITY OF CHICAGO,
Aug. 1, 1916.

* Lindelöf, Mathematische Annalen, vol. 2 (1870), p. 170.

A PROOF OF PETERSEN'S THEOREM.

By H. R. Brahana.

In the Acta Mathematica (Vol. 15 [1891], pp. 193–220) Julius Petersen proves the theorem that a primitive graph of the third degree has at least three leaves. I took the paper up in a series of reports in a Seminar on Analysis Situs and, following out suggestions made by Professor Veblen and Dr. Alexander, I have been able to get the proof into the present simpler form.

A *linear graph* is a set of 0-*cells** and 1-cells having the property that each 1-cell of the set is incident with two 0-cells of the set. The 0-cells are called the *vertices* of the graph.

A *regular graph* is a linear graph having the same number of 1-cells incident with each vertex. The *degree* of a regular graph is equal to the number of 1-cells at each vertex, and the *order* of the graph is equal to the number of vertices of the graph.

A regular graph is *factorable* when it contains a regular graph of the same order and of lower degree. We have as an obvious theorem,

A regular graph of degree k which contains a factor of degree k_1 also contains a factor of degree $k - k_1$.

For example, in Fig. 1 the four 0-cells A, B, C, D and the six 1-cells 1, 2, $\cdots$, 6 form a regular graph of order 4 and degree 3. It contains a factor of degree 1 (1-cells 5 and 6) and a factor of degree 2 (1-cells 1, 2, 3, and 4).

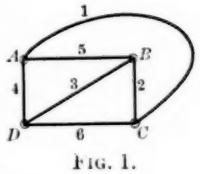

Fig. 1.

A *primitive* graph is a regular graph which can not be factored. Petersen proves that there exists no primitive graph of even degree greater than 2. On the other hand he gives a definite method by which we can construct a primitive graph of any odd degree.

A *leaf* is a portion of the graph which is connected with the remainder of the graph by a single 1-cell and which is such that no portion of it is connected with the remainder of it by a single 1-cell.

Petersen's theorem is as follows:

A primitive graph of degree 3 contains at least three leaves.

The theorem may be stated in the following form:

* Cf. Veblen and Alexander, *Annals of Mathematics*, vol. 14, p. 163. A 0-cell may be taken to be a point and a 1-cell to be an arc of curve (exclusive of its end-points) joining two points.

A regular graph of degree 3 which contains less than three leaves is factorable.

The proof will be made by a method of induction. Every third degree graph of order 2 satisfies the hypotheses of the theorem (as stated in the second form), and by constructing all of these graphs we see that the theorem holds. Let us assume that the theorem is not true; then from the fact that it is true for graphs of order 2 will follow that there exists a graph for which the theorem does not hold and of which the order is so low that for any graph of lower order the theorem does hold. We will show that this leads to a contradiction.

Suppose we have this minimum graph for which our theorem does not hold, and suppose that Fig. 2 represents a part of it. From this graph

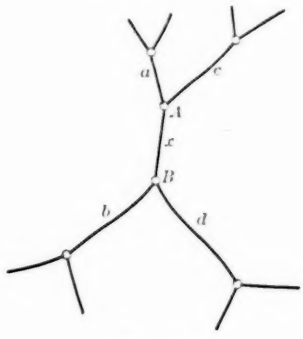

FIG. 2.

G we desire to get a new graph G' which will be of order less by 2 than the order of G and which will satisfy the hypotheses of our theorem. G' will then be factorable. We can always do this by removing a 1-cell x and coalescing a and c, the two remaining 1-cells at one of the vertices upon which x abuts, and b and d, the two 1-cells at the other vertex upon which x abuts, provided in so doing we do not obtain a graph with three leaves. To see that we can remove a 1-cell without introducing a third leaf we notice that if there is a leaf in G we may remove the 1-cell which connects the two parts of the graph and thus obtain two graphs, neither of which has three leaves, and if there is no leaf in G we may remove any 1-cell without introducing more than two leaves.

Now factor G' and color the 1-cells of the first degree factor red, those of the second degree factor blue. At each vertex of G' there are one red 1-cell and two blue 1-cells; if without destroying this situation we can get one red and two blue 1-cells at A and at B we will have factored G. There are three possibilities in the coloring of G': bd and ac may both be blue, (2) one may be red and the other blue, or (3) both may be red.

In case (1) we insert x colored red and have G factored. In the consideration of the other two cases we will use the phrase *necessarily red*, which when applied to one of the 1-cells ac and bd will mean that that 1-cell is red and that its color cannot be changed* without conflicting with the requirement that G' have one red and two blue 1-cells at each vertex or changing the other one of ac and bd from blue to red. We may assume that the one red 1-cell of case (2) and at least one of the two red 1-cells of case (3) are necessarily red, otherwise we could reduce each to case (1) and thus factor our graph G.

If either in case (2) or case (3), there exists a path of alternately red and blue 1-cells from A to B, we may permute the colors on that path and by inserting x colored blue we will have factored G. We have still to consider the case where one of the 1-cells ac and bd is necessarily red and no red-blue path from A to B exists. The argument that follows makes no assumption concerning the other of the two 1-cells ac and bd and consequently applies to both (2) and (3).

Suppose ac is necessarily red. From 1-cells and 0-cells of G' we will build a graph G'' which will be the set of all 1-cells and 0-cells that can be reached by red-blue paths starting with a or c and traversing no 1-cell more than once.

If all the red-blue paths which lead over a given 1-cell do so in the same direction, that 1-cell is *unicursal*; if there are paths traversing it in both directions it is *bicursal*.

Since there is no red-blue path from a to c a path starting with a may be continued until it runs into itself (i. e., until it reaches a vertex through which it has already passed). The last 1-cell of the path will be blue because there are two blue 1-cells at each vertex of G'. If the vertex at which the path ends was first reached on this path by a red 1-cell it is evident that the part of this path beginning and ending at this vertex may be traversed in either direction and that all of its 1-cells are bicursal. Shrink this circuit of bicursal 1-cells to a point; the resulting graph will still contain no path from a to c, for no new paths are introduced. At every vertex there will be one and only one red 1-cell; at the new vertex there may be any even number of blue 1-cells.† We continue this shrinking process as long as there are such bicursal circuits. When all

* Whenever a 1-cell of G' is on a closed circuit of alternately red and blue 1-cells we may permute the colors on that circuit, thus changing the color of the 1-cell, and each vertex of G' will still have one red and two blue 1-cells.

† Only blue 1-cells (with the exception of the one red 1-cell) abut upon this vertex for the red 1-cell at every vertex on the bicursal circuit is contained in the circuit and is therefore removed by the shrinking. There are an odd number of vertices on such a circuit and consequently an odd number of 1-cells at the new vertex which results from the shrinking; of these 1-cells, an even number must be blue.

such circuits have been shrunk to single 0-cells it is evident that there are no bicursal 1-cells in the new graph for a bicursal 1-cell would imply a red-blue path from a to c, which does not exist. Thus the new graph, which we will call G''', will be made up of unicursal 1-cells; there will be one red 1-cell at each vertex. Each of the new vertices of G''' (those obtained by the shrinking process) will have an even number of blue 1-cells, which number may sometimes be zero. A 0-cell at which there are no blue 1-cells will represent a leaf of G, for the graph G can be separated into two parts by severing the red 1-cell that leads to that 0-cell. Each vertex of G'' at which there is only one unicursal blue 1-cell * will be a vertex of G''' and at such a vertex there will be but one blue 1-cell.

Now let

b_0 = the number of vertices of G''' with no blue 1-cells (= the number of leaves in G'''),
b_1 = the number of vertices of G''' with 1 blue 1-cell,
b_2 = the number of vertices of G''' with 2 blue 1-cells,
. . . . . .
$b_{2\alpha}$ = the number of vertices of G''' with 2α blue 1-cells,

where 2α is the maximum of the number of blue 1-cells at vertices of G''', and let

R = the number of red 1-cells of G''',
B = the number of blue 1-cells of G''',
P = the number of vertices of G'''.

We will not count the 1-cells a and c in G'''.

Now if we let $D = \sum_{i=0}^{\alpha} 2(1-i)b_{2i}$, which gives us the "deficiency" of 1-cells at vertices of G''' if we agree that normally there are three 1-cells at each vertex except those at which there is only one blue 1-cell of G'''; and if we then count the ends of 1-cells at vertices of G''', we get the following equation:

$$3P - D - b_1 = 2R + 2B + 2. \tag{1}$$

Counting the vertices, we get the equation:

$$P = 2R + 2. \tag{2}$$

Counting the blue 1-cells, we get the equation:

$$B = 2R + 4 - D. \tag{3}$$

We see (3) from the fact that we may go from a red 1-cell to two blue

* A red-blue path can approach such a vertex only by a blue 1-cell and leave it only by a red 1-cell.

1-cells at every vertex we can reach by a red 1-cell excepting those that are accounted for by D; the 4 enters because the 1-cells a and c are not part of G'''.

From these three equations we get

$$D = 4 + b_1. \tag{4}$$

It is obvious that D is positive and b_1 is even. The only positive term of the sum D is the one for which $i = 0$; from this fact and (4) we get

$$2b_0 \geqq 4 + b_1. \tag{5}$$

Thus for $b_1 \geqq 2$ we have $b_0 \geqq 3$. For $b_1 = 0$, $b_0 \geqq 2$; but in this case (since $b_1 = 0$) the only connection between G'' and the remainder of G is the 1-cell x and therefore the remainder contains at least one leaf, which with the two leaves of G''' gives us the required three. This completes the proof of Petersen's theorem.

PRINCETON, N. J.

COMPLETE EXISTENTIAL THEORY OF BERNSTEIN'S SET OF FOUR POSTULATES FOR BOOLEAN ALGEBRAS.*

By J. S. Taylor.

A set of m postulates is considered independent in the ordinary sense if no one of the m postulates is implied by the others. Professor E. H. Moore,† however, has suggested the problem, in connection with sets of postulates, of determining not only the implicational relations existing among the postulates as they stand but also all the implicational relations which exist among properties defined either by the postulates themselves or by the negatives of the postulates. A set of postulates is then said to be *completely independent* if, and only if, no such implicational relations exist. For example, Professor L. L. Dines‡ has shown that while Sheffer's five postulates for Boolean algebras are independent in the ordinary sense, they are not *completely independent* in the sense defined by Professor Moore, since the negative of the first postulate implies the third, fourth, and fifth.

Bernstein's§ set of four postulates for the same discipline is the most concise set yet formulated for Boolean algebras. The determination of all implicational relations existing among these postulates and their negatives is therefore of corresponding interest and simplicity. The four postulates in question define properties of a system Σ in which the primitive ideas are the class K and the binary rule of combination $|$ between elements of K. Any system $\Sigma(K, |)$ of the prescribed type has with respect to the four postulates one of the $2^4 = 16$ characters:

$$(1) \qquad (++++), \quad (+++-), \quad \cdots \quad (+---), \quad (----);$$

the ith sign of the character being plus or minus according as Σ has or has not the ith property. The body of sixteen propositions stating for the various characters represented in (1) that there exists or does not exist a system having the character in question constitutes what Professor Moore has called "the complete existential theory" of the four postulates.

* Read before the San Francisco Section of the American Mathematical Society, November 25, 1916.

† E. H. Moore, "Introduction to a Form of General Analysis," New Haven Mathematical Colloquium, Yale University Press, page 82.

‡ L. L. Dines, "Complete Existential Theory of Sheffer's Postulates for Boolean Algebras," Bulletin, vol. XXI, Number 4, pp. 183–188.

§ B. A. Bernstein, "A Set of Four Independent Postulates for Boolean Algebras," Transactions, Vol. 17 (1916), pp. 50–52.

1. Bernstein's postulates. Their complete existential theory. Bernstein's postulates concerning systems $\Sigma(K, |)$ are:

P_1. *K contains at least two distinct elements.*

P_2. *If a, b are elements of K, $a|b$ is an element of K.*

Definition 1. $a' = a|a$.

P_3. *If a, b, and the combinations indicated are elements of K,*

$$(b|a)|(b'|a) = a.$$

P_4. *If a, b, c, and the combinations indicated are elements of K,*

$$a'|(b'|c) = [(b|a')|(c'|a')]'.$$

For the four postulates as given the complete existential theory consists of six propositions of non-existence and ten propositions of existence. In particular, postulates P_2, P_3, and P_4 are completely independent. The six non-existences arise from the fact that the negative of P_1 implies P_3 and P_4.

Propositions of Non-Existence.

The six propositions of non-existence, as implied above, may be expressed by the single proposition:

$$\Sigma^{-1} \supset \Sigma^{34}. \tag{2}$$

That is: "If a system Σ does not satisfy P_1, then it does satisfy P_3 and P_4."

The truth of this statement is readily perceived from the following considerations. The hypothesis that P_1 is not satisfied necessitates either K^{null} (a class without any elements) or K^{singular} (a class with only one element). But if K contains no elements, P_3 and P_4 are satisfied vacuously. And if K contains only one element, then P_3 and P_4 are satisfied either evidently or vacuously, according as Σ does or does not satisfy P_2.

Proposition (2) renders impossible the existence of systems with the following six characters:

$$(-++-);\quad (-+--+);\quad (-+---);$$
$$(--+-);\quad (---+);\quad (----).$$

Propositions of Existence.

The ten propositions of existence are established by the exhibition of ten systems having the ten remaining characters; there are two examples for K^{singular}, seven for K^{dual}, and one for K^{triple}. In each case K contains the least number of elements possible.

Examples for K Singular.

Systems having the characters $(-+++)$ and $(--++)$ respectively are the following:

System I_1. Character $(-+++)$; class composed of single element e_1 with $e_1|e_1 = e_1$.

System I_2. Character $(--++)$; class the same as above with $e_1|e_1 \neq e_1$.

Examples for K dual.

In the examples following $e_i|e_j$ will be defined by means of tables. For instance, if K is a class of two elements, e_1 and e_2, and if $e_1|e_1 = e_1$, $e_1|e_2 = e_1$, $e_2|e_1 = e_2$, and $e_2|e_2 = x$ (x not a member of K), this will be expressed by the table:

	e_1	e_2
e_1	e_1	e_1
e_2	e_2	x

Throughout the discussion x will mean an element not in K.

System II_1.

$(++++)$

	e_1	e_2
e_1	e_2	e_1
e_2	e_1	e_1

System II_2.

$(+++-)$

	e_1	e_2
e_1	e_1	e_2
e_2	e_1	e_2

System II_3.

$(++-+)$

	e_1	e_2
e_1	e_1	e_1
e_2	e_1	e_1

System II_4.

$(++--)$

	e_1	e_2
e_1	e_1	e_1
e_2	e_2	e_2

System II_5.

$(+-++)$

	e_1	e_2
e_1	x	x
e_2	x	x

System II_6.

$(+--+)$

	e_1	e_2
e_1	x	e_2
e_2	e_2	e_2

System II_7.

$(+---)$

	e_1	e_2
e_1	e_1	e_2
e_2	x	e_1

Example for K triple.

System III_1.

$(+-+-)$

	e_1	e_2	e_3
e_1	e_1	e_2	x
e_2	e_1	e_2	x
e_3	x	x	x

That postulates $P_2 - P_4$ are completely independent follows from the fact that systems have been exhibited having the $2^3 = 8$ characters $(+\pm\pm\pm)$.

2. **Bernstein's postulates made completely independent.**

Now, although Bernstein's postulates have thus been proved to be not *completely independent*, they become so if postulate P_1 is replaced by the following proposition:

P_1'. *K contains at least four distinct elements.*

The complete independence of the modified set of postulates is proved by the exhibition of sixteen systems having the sixteen different characters of (1). Whenever possible, systems whose tables have already been given will be used and will be referred to merely as I_1, I_2, etc.

Examples for K singular.

$(-+++)$, I_1; $\quad$ $(--++)$, I_2.

Examples for K dual.

$(-+++-)$, II_2; $\quad$ $(-+-+)$, II_3; $\quad$ $(-+--)$, II_4; $\quad$ $(---+)$, II_6; $(----)$, II_7.

Example for K triple.

$(--+-)$, III_1.

Examples for K quadruple.

System IV_1.

$(++++)$

	e_1	e_2	e_3	e_4
e_1	e_2	e_1	e_4	e_3
e_2	e_1	e_1	e_1	e_1
e_3	e_4	e_1	e_4	e_1
e_4	e_3	e_1	e_1	e_3

System IV_2.

$(+++-)$

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_1	e_2	e_3	e_4
e_3	e_1	e_2	e_3	e_4
e_4	e_1	e_2	e_3	e_4

System IV_3.

$(++-+)$

	e_1	e_2	e_3	e_4
e_1	e_1	e_1	e_1	e_1
e_2	e_1	e_1	e_1	e_1
e_3	e_1	e_1	e_1	e_1
e_4	e_1	e_1	e_1	e_1

System IV_4.

$(++--)$

	e_1	e_2	e_3	e_4
e_1	e_1	e_1	e_1	e_1
e_2	e_2	e_2	e_1	e_1
e_3	e_1	e_1	e_1	e_1
e_4	e_1	e_1	e_1	e_1

System IV_5.

$(+-++)$

	e_1	e_2	e_3	e_4
e_1	x	x	x	x
e_2	x	x	x	x
e_3	x	x	x	x
e_4	x	x	x	x

System IV_6.

$(+--+)$

	e_1	e_2	e_3	e_4
e_1	x	e_2	e_2	e_2
e_2	e_2	e_2	e_2	e_2
e_3	e_2	e_2	e_2	e_2
e_4	e_2	e_2	e_2	e_2

System IV_7.

$(+---)$

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	x	x
e_2	x	e_1	x	x
e_3	x	x	x	x
e_4	x	x	x	x

System IV_8.

$(+-+-)$

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	x	x
e_2	e_1	e_2	x	x
e_3	x	x	x	x
e_4	x	x	x	x

That Bernstein's set of four postulates cannot be made completely independent by postulating a minimum of less than four elements may readily be proved. The results obtained in § 1 clearly show that the postulates will not become completely independent if a minimum less than three is postulated. There remains, then, the possibility of postulating a minimum of exactly three elements. If the postulates then become completely independent, it must be possible to devise systems having all the sixteen characters of (1). But this is not the case, for it is then impossible for a system of the character $(---+-)$ to exist. This the reader can verify without serious difficulty.*

* The proof of this may be outlined as follows. Suppose P_1' be replaced by P_1'': *K contains at least three distinct elements.* Then if a system of the character $(---+-)$ is to exist, there must be a class K, null, singular, or dual, which contradicts P_2 and P_4 while it satisfies P_3. But it has been proved above that K-null or K-singular implies P_4. Let us then consider K to consist of two elements, e_1 and e_2. Since P_2 must be contradicted, at least one of the four combinations $e_i|e_j$ $(i, j = 1, 2)$ must be an element x not in K. But (1) if $e_1|e_1 = x$ or $e_2|e_2 = x$, P_4 is satisfied (vacuously or evidently), as can readily be verified; (2) if $e_1|e_2 = x$, it is found, by letting $e_1|e_1$, $e_2|e_2$, and $e_2|e_1$ take on all possible values, that we get systems which contradict P_3 or which satisfy P_4, but none of the character $(---+-)$; and (3) if $e_2|e_1 = x$, the results are the same as for the case $e_1|e_2 = x$. No system of the character $(---+-)$ exists under the postulation of P_1'', and the corresponding set of postulates is therefore not completely independent.

It is interesting to note in this connection that the fact that the number of elements in a

The proposed substitution of P_1' for P_1 is, therefore, the simplest alteration which will make the postulates completely independent. As far as the writer knows, Bernstein's set of postulates as thus modified is the only set of postulates for Boolean algebras whose complete independence has been proved.

UNIVERSITY OF CALIFORNIA,
March, 1917.

finite class satisfying Boolean algebras is a power of 2 does not help us in settling the question of complete independence. While one can deduce by its use the fact that a class containing at least three elements necessarily contains at least four, so that Bernstein's set of postulates wherein P_1 is replaced by P_1'' is equivalent to that set when P_1 is replaced by P_1', it does *not* follow from this that if one set is completely independent, the other one is. Equivalence merely means that each set can be deduced from the other. It does not imply any other relationship between the two sets.

RATIO, PROPORTION AND MEASUREMENT IN THE ELEMENTS OF EUCLID.

By Henry B. Fine.

The following note is concerned with a few of the definitions and theorems of the fifth, sixth, seventh and tenth books of the Elements of Euclid, all of them relating to the theory of measurement, ratio and proportion. The fifth book, it may be explained, is devoted to the theory of ratio and proportion of magnitudes in general and the sixth to the applications of this theory in plane geometry, the seventh is the first of Euclid's three arithmetical books, and the tenth an elaborate treatise on incommensurable magnitudes.

According to tradition it was Pythagoras who first proved that the side and diagonal of a square are incommensurable. The early discovery of incommensurability had important consequences for the Greek mathematics. It revealed the fact that the theory of proportion of numbers and other commensurables already known to the Pythagoreans was inadequate for geometry and led to the invention of a general theory—that of Euclid's fifth book—which is independent of commensurability, and, as a consequence of this, to the complete exclusion of the notion of numerical measurement from the Greek geometry. But it also started investigations which resulted in the development of the doctrine of incommensurable magnitudes—chiefly line segments and rectangles corresponding to quadratic irrationalities—into a department of mathematics coördinate with arithmetic and geometry. The commentator Proclus attributes the general theory of proportion to Eudoxus and much of the doctrine of incommensurables to Theaetetus. Both lived in the century before Euclid (the fourth century B.C.).

Both of the above mentioned theories of proportion are contained in Euclid's Elements. The general theory is that presented in the fifth book and applied in the subsequent geometrical books. The older theory is developed in the seventh book as a theory for numbers but is used in the tenth book as applicable to all classes of commensurables. There is no intimation that this theory is included in the other. It is characteristic of both theories, as Euclid presents them, that they are based on a definition not of ratio but of proportion.

It may be added that the number system of Euclid is restricted to

the cardinal numbers. He defines a number as a "multitude composed of units." In the proofs of the theorems in the arithmetical books numbers are represented by line segments in the same way that magnitudes in general are represented in the fifth and tenth books. The association of the notion of geometric magnitude with number is seen also in the characterization of numbers as plane, solid, similar, square, cube, and in modes of expression often used in proofs of theorems.

The seventh book begins by showing that if any two numbers be given they may be proved prime to one another, or their greatest common divisor be found, by the process which we now call the Euclidean method. From this result the conclusion is drawn that any given number is a definite determinable multiple, part, or "parts" (multiple of a part) of any other given number, and the theory of proportion of numbers is then developed on the basis of the definition:

Numbers are proportional when the first is the same multiple, part, or parts of the second that the third is of the fourth.

By thus basing the theory on the equality of ratios instead of ratio itself, the Greeks were enabled to discuss the fractional relation without the use of the fraction.

The tenth book opens with the theorem:

If from the greater of two given magnitudes of the same kind there be taken its half or more, from the remainder its half or more, and so on, a remainder will at length be reached which is less than the smaller of the two given magnitudes (*X*, 1).

Special interest attaches to this theorem as being that on which the method of exhaustions is based. The theorem itself and the assumption on which its proof is made to depend—the so-called "Axiom of Archimedes," that of the smaller of two given magnitudes of the same kind a multiple can be found which will exceed the greater—are probably both due to Eudoxus, for Archimedes attributes the method of exhaustions to this mathematician. Next comes the fundamental theorem:

Let A and B denote two unequal magnitudes of the same kind, of which A is the greater. Suppose B to be repeatedly subtracted from A until a remainder R_1 is found which is less than B, then R_1 to be repeatedly subtracted from B until a remainder R_2 is found which is less than R_1, and so on. If, however far this process may be carried, a remainder will never be found which is contained exactly in the remainder immediately preceding it, then the magnitudes A and B are incommensurable (*X*, 2).

The proof may be summarized thus: If A and B have any common measure, let it be E. Then E will also be a measure of all the remainders $R_1, R_2, \cdots$. But R_1 is less than half of A, R_3 less than half of R_1 and so

on, and similarly R_2 is less than half of B, R_4 less than half of R_2, and so on. Hence, by the preceding theorem, all the remainders after a certain one will be less than E and cannot therefore be measured by E. The theorem follows from this contradiction.

From the proof of this theorem it follows that: *If on the contrary the magnitudes A and B are such that the process above described when applied to them will ultimately yield a remainder R_n which is exactly contained in that immediately preceding it, then A and B are commensurable and R_n is their greatest common measure* (X, 3).

A basis is thus secured for the proof of the following theorem, which in the rest of the tenth book plays the rôle of a working definition of commensurable magnitudes:

Commensurable magnitudes have to one another the ratio which a number has to a number (X, 5).

The proof is this: If A and B denote the magnitudes and D their greatest common measure, we shall have $A = aD$ and $B = bD$, where a and b are integers. Therefore, by the definition of proportion above given (here assumed by Euclid as a definition for commensurable magnitudes in general), $A : D :: a : 1$ and $D : B :: 1 : b$. Hence (VII, 14) $A : B :: a : b$.

One more proposition of this book, a consequence of that just proved, may be cited, the substance of which is that: *The squares on commensurable line segments have to one another the ratio which a square number has to a square number; and therefore line segments the squares on which have not such a ratio are incommensurable* (X, 9). It is attributed to Theaetetus and is of interest as being the generalization of the theorem that the side and diagonal of a square are incommensurable, in which this entire doctrine had its origin, and as being the first proof of the actual existence of incommensurables which occurs in Euclid's Elements.

One who examines Euclid's presentation of this theory of proportion can but be impressed by the care he takes before applying his definition to provide a general test—the method of greatest common measure—for actually determining for any given numbers or other commensurables whether or not they satisfy its conditions. One is reminded of the dictum of Kronecker that no mathematical definition can be regarded as resting on a secure foundation unless means are at hand for determining by a finite number of steps whether any given object conforms to it or not. This was the Greek doctrine of mathematical definition. Aristotle says: "In geometry the existence of points and lines is assumed, the existence of everything else must be proved, and nothing may be legitimately used whose existence has not been proved." It is for this reason also that

Euclid was at such pains to avoid hypothetical constructions in his geometric books.

Let us now turn to the general theory of proportion of the fifth book. The significant definitions of the book are these:*

DEFINITION 3. *A ratio is a sort of relation in respect of size between two magnitudes of the same kind.*

DEFINITION 4. *Magnitudes are said to have a ratio to one another which are capable when multiplied of exceeding one another.*

DEFINITION 5. *Let A, B, X, Y be four magnitudes, A of the same kind as B, X of the same kind as Y. If for all integral values of m and n t be the case that according as $mA \gtreqless nB$, so also is $mX \gtreqless nY$, then A is said to be in the same ratio to B as X to Y.*

DEFINITION 6. *Four magnitudes A, B, X, Y related as in Definition 5 are called proportional.*

DEFINITION 7. *If* (in the notation of Definition 5) *m and n can be found such that $mA > nB$ but $mX \not> nY$, then A is said to have a greater ratio to B than X has to Y.*

Here, as in the case of the earlier or arithmetical theory just considered, there is no explicit definition of ratio. Definition 3 is merely a vague indication of the sort of thing that ratio is and Definition 4 is equivalent to a statement that ratio exists between any two "magnitudes" for which the axiom of Archimedes (Eudoxus) above cited holds good. But a very definite notion of ratio is implied in the definition of proportion. For according to Definition 5 the condition that A, B, X, Y be proportional is this: If the multiples $A, 2A, 3A, \cdots$ and $B, 2B, 3B, \cdots$ be supposed arranged in a single sequence in the order of size, and so likewise the multiples $X, 2X, 3X, \cdots$ and $Y, 2Y, 3Y, \cdots$, the law of distribution of the multiples of A among those of B must be the same as that of the multiples of X among those of Y. Hence the "sameness" of the ratios $A : B$ and $X : Y$ means the sameness of these two laws of distribution, and the ratio $A : B$ itself means that size relation between A and B which is indicated by the manner in which the multiples of A are distributed among those of B. A single ratio does not admit of representation in finite terms. But for a theory of proportion and an abstract theory of ratio in general the representation of single ratios is not requisite. Sound and applicable definitions of the relations of equality and greater and lesser inequality for ratios are all that is needed. These are provided in Definitions 5 and 7. That in brief is the point of view which this scheme of definitions implies.

*I have quoted Definitions 3, 4 from Heath's Euclid, vol. 2, p. 114, and have paraphrased Definitions 5–7.

For synthetic geometry no simpler or more elegant definition of proportion than Definition 5 can be desired. Thus, consider Euclid's proof of the theorem that: *Triangles which have the same altitude are to one another as their bases* (*VI*, 1). If T_1, T_2 be the triangles and b_1, b_2 their bases, and any multiples of b_1, b_2 be taken, say mb_1, nb_2, and the corresponding multiples of T_1, T_2, namely mT_1, nT_2, it follows from the simplest geometrical considerations that according as $mb_1 \gtreqless nb_2$ so also is $mT_1 \gtreqless nT_2$. Hence $b_1 : b_2 :: T_1 : T_2$.

The mention of a few of the theorems of Book V will suffice to show how Euclid deduces the properties of proportions and the elements of an abstract theory of ratio from his definitions.*

THEOREMS 7–10. *According as* $A \gtreqless B$ *so also is* $A : C \gtreqless B : C$ *and* $C : A \lesseqgtr C : B$; *and conversely.*

Thus, if $A > B$, then $A : C > B : C$. For take m so that $m(A-B) > C$ and $mB > C$ (Definition 4). Next take n so that $(n-1)C \leqq mB < nC$. Then $mA > nC$ but $mB < nC$. Hence $A : C > B : C$ (Definition 7).

THEOREM 11. *If* $A : B :: K : L$ *and* $K : L :: X : Y$, *then* $A : B :: X : Y$.

For, by Definition 5, according as $mA \gtreqless nB$, so also is $mK \gtreqless nL$ and therefore $mX \gtreqless nY$. Hence $A : B :: X : Y$.

THEOREM 13. *If* $A : B :: K : L$ *and* $K : L > X : Y$, *then* $A : B > X : Y$.

For, by Definitions 7, 5, m and n can be so chosen that $mK > nL$ but $mX \ngtr nY$, and therefore that $mA > nB$ but $mX \ngtr nY$. Hence $A : B > X : Y$.

THEOREM 16. *If* A, B, C, D *are of the same kind, and* $A : B :: C : D$, *then, "alternately,"* $A : C :: B : D$.

For it readily follows from Definition 5 that $mA : mB :: nC : nD$. By Theorems 7–10, 11, 13, according as $mA \gtreqless nC$, so also is $mA : mB \gtreqless nC : mB$, $\therefore nC : nD \gtreqless nC : mB$, $\therefore mB \gtreqless nD$. Hence $A : C :: B : D$ (Definition 5).

THEOREM 18. *If* $A : B :: X : Y$, *then* $A + B : B :: X + Y : Y$.

For if $m \geqq n$, then $m(A+B) > nB$ and $m(X+Y) > nY$. If $m < n$, then, according as $m(A+B) \gtreqless nB$, so also is $mA \gtreqless (n-m)B$, $\therefore mX \gtreqless (n-m)Y$, $\therefore m(X+Y) \gtreqless nY$. Hence, by Definition 5, $A + B : B :: X + Y : Y$.

THEOREM 22. *If* $A : B :: X : Y$ *and* $B : C :: Y : Z$, *then "ex aequali"* $A : C :: X : Z$.

For it readily follows from Definition 5 that $mA : nB :: mX : nY$ and $nB : pC :: nY : pZ$. According as $mA \gtreqless pC$, so also is $mA : nB \gtreqless pC : nB$, $\therefore mX : nY \gtreqless pZ : nY$, $\therefore mX \gtreqless pZ$. Hence $A : C :: X : Z$.

* Euclid represents the magnitudes and their multiples by line segments. In other respects the proofs as here stated, except that of Theorem 18, are substantially as Euclid gives them.

THEOREM 24. *If both* $A : C :: X : Z$ *and* $B : C :: Y : Z$, *then also* $A + B : C :: X + Y : Z$.

For since $A : C :: X : Z$ and $C : B :: Z : Y$ (Definition 5), we have $A : B :: X : Y$, $\therefore A + B : B :: X + Y : Y$, $\therefore A + B : C :: X + Y : Z$ (Theorems 18, 22).

Theorem 22 implies a definition of the product of the two ratios $A : B$ and $B : C$, and Theorem 24 a definition of the sum of the two ratios $A : C$ and $B : C$, in the same sense that Definition 5 implies a definition of ratio itself. The Greeks called $A : C$ the ratio "compounded" of the ratios $A : B$ and $B : C$ or of any two ratios equal to these. Thus in VI, 23, Euclid proves that: *Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.*

There is a close relationship between the notion of the ratio of two incommensurables implied in Definition 5 and the irrational number as defined by Dedekind. The essential element in Definition 5 is the recognition of the fact that when A and B are incommensurable a definition of the ratio $A : B$ involves the comparison of mA and nB for all pairs of positive integral values of m and n, and the separation of these pairs into two classes, the class (m_1, n_1) for which $m_1A > n_1B$ and the class (m_2, n_2) for which $m_2A < n_2B$. This separation of all integral pairs (m, n) into the two classes (m_1, n_1) and (m_2, n_2) is identical with the Dedekind cut defining the irrational number which expresses the ratio $A : B$. Moreover Euclid was in possession of everything needed to give the notion of this cut the same expression in terms of the ratios $n_1 : m_1$ and $n_2 : m_2$ that the Dedekind definition would give it in terms of the fractions n_1/m_1 and n_2/m_2. For since $m_1A > n_1B$ and $m_2A < n_2B$, we have $m_1n_2 > n_1m_2$ and therefore $n_1 : m_1 < n_2 : m_2$, by Definition 7. Furthermore, by what is contained in the proof of X, 2, cited above, if any particular pair m_1', n_1' be assigned, another pair m_1, n_1 can be found such that $m_1'A - n_1'B > m_1A - n_1B$ and $n_1 > n_1'$ and therefore such that $n_1(m_1'A - n_1'B) > n_1'(m_1A - n_1B)$, or $n_1m_1' > n_1'm_1$, or $n_1 : m_1 > n_1' : m_1'$ (Definition 7); hence there is no greatest $n_1 : m_1$. And similarly there is no least $n_2 : m_2$. Hence, had Euclid created numbers to correspond to his ratios $n_1 : m_1$ and $n_2 : m_2$, he would have been in a position to create an irrational number to correspond to the ratio $A : B$ and to define this number ordinally with respect to the numbers $n_1 : m_1$ and $n_2 : m_2$ quite after the manner of Dedekind.

Since the natural numbers are included among the ratios $n : m$, and since $n_1 : m_1 < A : B < n_2 : m_2$ (as readily follows from Definition 7), the system of ratios, commensurable and incommensurable, as defined by Euclid, is in fact a part of the system of real numbers as we define that system, a part which becomes identical with the whole when the postulate

is made that for every Dedekind cut there exists a pair of magnitudes A and B which will yield this cut in the manner above described (Cantor's axiom). Of course, Euclid had no such conception of his ratios; even the concept of $n : m$ as a single number was foreign to him.

It was merely as useful symbols in algebraic reckoning and without definition that irrational numbers first made their way into mathematics. The irrational was not defined until the renascence in mathematics of the fine critical insight possessed by the Greeks. The basis of sound definition was then found in notions already familiar to Eudoxus and Euclid.

Euclid avoids the fraction and the irrational by basing his theory of proportion upon the equality of ratios instead of ratio itself, and his definition is in terms of the indeterminate integral multipliers m and n. One who has read the writings of Kronecker—as extreme a purist with respect to number as the Greeks themselves—will notice an interesting parallelism between this procedure of Euclid and the way of escape from algebraic numbers found by Kronecker in the use of congruences involving indeterminates.*

The importance of the rôle which this general theory of proportion played in the Greek mathematics is not easily overstated. It was the key to the solution of the problem set by the discovery of incommensurability, the creation of a theory of geometric magnitude independent of the notion of numerical measurement. But it was also the instrument which for the Greek geometers served the purposes which are served for us by an algebraic algorithm and the equation. It was an unwieldy instrument but they developed great power in the use of it, as is shown for example in Euclid's geometric treatment of the quadratic (VI, 27–29) and in his discussion and classification of binomial line segments corresponding to quadratic irrationalities in the tenth book, but more strikingly still in the Conics of Apollonius.

PRINCETON, N. J.

* L. Kronecker *Ueber den Zahlbegriff*. Journal für die reine und ange wandte Mathematik Band 101.

FACTORIZATION OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES.

By William F. Osgood.

The theorems of factorization with which this paper deals are in part so obvious as to seem trivial. In part, however, they involve definitions and conceptions, the necessity for which is not evident at a glance, nor can their justification be established on any *a priori* grounds; one has to go over the whole field and ascertain what the situation is that is to be met.

The main theorems are contained in a memoir by Weierstrass (cf. § 3 below); but the presentation in that paper is altogether inadequate for making clear to the reader that which is essential.

This theory of factorization is of much importance in practice, and it is embarrassing for a writer, in studying a subject in the theory of analytic functions of several complex variables, to be obliged to assume it, well knowing that most readers have never thought it through in detail. It is for this reason that I have ventured to offer a systematic presentation of the theory as a whole.*

The reader will do well to turn at once to § 4, for he will find there one of the chief topics with which the paper deals.

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§ 1. Cauchy's Contribution.

Definitions. A function $F(z_1, \cdots, z_n)$ of several complex variables is said to be *analytic in* or *at* a point $(z_1, \cdots, z_n) = (a_1, \cdots, a_n)$—or more briefly $(z) = (a)$—if it is possible to embed each coördinate a_k in a region T_k of the complex z_k-plane, these regions being chosen in such a manner that

* A paper by Dautheville, Annales de l'École Normale Supérieure, (3) 2 (1885) supplement, is also devoted to an exposition of this subject.

(*a*) F is defined at each point (z), where z_k is an arbitrary point of T_k, $k = 1, \cdots, n$;

(*b*) the n partial derivatives

$$\frac{\partial F}{\partial z_k}, \qquad k = 1, \cdots, n,$$

all exist at every such point;

(*c*) these derivatives are all continuous at all the points in question.

It has been shown that condition (*c*) is a consequence of conditions (*a*) and (*b*); but the proof is not elementary, and does not belong at the beginning of the theory.

If F is analytic at the point $(z) = (a)$, then F can be developed by Taylor's Theorem about the point (a).

The domain of the points $(z_1, \cdots, z_n)$ in which a function F is considered may be any $2n$-dimensional region T of the real space of the $2n$ real variables $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$ where

$$z_k = x_k + iy_k.$$

The function F is said to be *analytic in* T if it is analytic at each point of T.

A special class of regions T of use in practice consists of the *cylindrical regions*, i. e., regions such that each z_k is, as in the above definition, an arbitrary point of a fixed region T_k of the z_k-plane, $k = 1, \cdots, n$.

Cauchy's Analysis. Let $F(w, z)$ be a function of the two complex variables w, z, which is analytic in the point (b, a), and let

$$F(b, a) = 0, \qquad F(w, a) \not\equiv 0.$$

Then $F(w, a)$ will have a root of the mth order in $w = b$, where $m \geqq 1$; and it will have no second root in the neighborhood of the point $w = b$.

Let

$$|w - b| < r_1, \qquad |z - a| < h_1$$

be a region in which F is analytic. Then it is possible to choose a positive $r < r_1$ so that $F(w, a)$ does not vanish at any point of the region $0 < |w - b| \leqq r$:

$$F(w, a) \neq 0, \qquad 0 < |w - b| \leqq r.$$

Let W be an arbitrary point of the circle

$$|W - b| = r.$$

We can then find a positive $h \leqq h_1$ such that $F(W, z)$ will not vanish for any z of the circle $|z - a| < h$:

$$F(W, z) \neq 0, \qquad \text{if} \qquad |W - b| = r, \qquad |z - a| < h.$$

For, the points (w, z) of the four-dimensional space of these variables, in which $w = W$ and $z = a$, i. e., the points

$$|W - b| = r, \qquad z = a,$$

form a real curve or *chain*—more specifically a *connected perfect manifold*—in each point of which $F(w, z)$ is continuous and different from 0. It must, therefore, be possible to embed this manifold in a four-dimensional neighborhood, at each point (w, z) of which $F(w, z)$ is different from 0, and clearly this neighborhood can be chosen as cylindrical. If T_2 is the corresponding region of the z-plane, then h can be taken as the shortest distance from $z = a$ to a boundary point of T_2—or as any smaller positive quantity. For convenience, we will henceforth take $b = 0$.

And now I say: *If z be an arbitrary point of the region*

S: $$|z - a| < h,$$

*then $F(w, z)$, regarded as a function of w alone, has precisely m roots in the region**

K: $$|w| < r.$$

In fact, for such a value of z, $F(w, z)$ has no root on the circle

C: $$|W| = r,$$

and the number of roots within the circle is given by the formula

$$\frac{1}{2\pi i}\int_C \frac{F_w(W, z)dW}{F(W, z)}, \qquad F_w(w, z) = \frac{\partial F}{\partial w},$$

where the integral is extended in the positive sense over C. In the point $z = a$ this integral has the value m. Moreover, it is a continuous function of z, and being an integer, it must, therefore, remain constant.

Let $\varphi(w)$ be any function of w which is analytic within C and continuous on the boundary. Then the integral

$$\frac{1}{2\pi i}\int_C \varphi(W)\frac{F_w(W, z)dW}{F(W, z)},$$

extended in the positive sense over the circle C, has for its value†

$$\varphi(w_1) + \cdots + \varphi(w_m) = P(z),$$

where $w_1, \cdots, w_m$ are the m roots of $F(w, z)$. The function $P(z)$ is seen from inspection of the integral to be analytic within C.

* Cauchy, Lithographed Turin memoir of October 11, 1831 = Execices d'analyse, vol. 2 (1841), p. 64.

† Cauchy, l. c., p. 66.

If, in particular, $m = 1$ and we set $\varphi(w) = w$, the above equation becomes

$$w = P(z),$$

and thus the roots w of $F(w, z)$ which lie in the circle C when z is a point of S form a function which is analytic in S. We have here, probably, the first rigorous proof of an existence theorem for an implicit function in a general case.*

So far Cauchy carried his analysis in the year 1831. We turn now to the further development due to Weierstrass in the sixties.

§ 2. A General Theorem on Implicit Functions.

If, in the preceding paragraph, we set† $\varphi(w) = w^k$, we find:

$$w_1^k + \cdots + w_m^k = P_k(z),$$

where $P_k(z)$ is analytic in the circle S. Let

$$\begin{aligned} A_1 &= -(w_1 + \cdots + w_m), \\ A_2 &= w_1w_2 + w_1w_3 + \cdots + w_{m-1}w_m, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_m &= (-1)^m w_1 \cdots w_m. \end{aligned}$$

Then A_k can be expressed as a polynomial in $P_1(z), \cdots, P_m(z)$, and hence is a function of z analytic in S.

The foregoing analysis applies with mere formal modifications to the case of a function $F(w, z_1, \cdots, z_n)$, and thus we obtain the following

THEOREM. *Let $F(w, z_1, \cdots, z_n)$ be analytic in the point $(b, a_1, \cdots, a_n)$, and let*

$$F(b, a_1, \cdots, a_n) = 0, \qquad F(w, a_1, \cdots, a_n) \not\equiv 0.$$

For simplicity, let $b = 0$.

There exists then a definite neighborhood K of the point $w = 0$ and a neighborhood S of the point $(a_1, \cdots, a_n)$,

K: $$|w| < r;$$

S: $$|z_k - a_k| < h_k, \qquad k = 1, \cdots, n,$$

such that, if (z) be an arbitrary point of S, the equation

$$F(w, z_1, \cdots, z_n) = 0$$

has precisely m roots in K.

* Cauchy, l. c., p. 71.

† The form of the proof of Weierstrass's Theorem of § 3 which begins with this substitution is due to Simart. Cf. Picard, Traité d'analyse, vol. 2, ch. 9.

Moreover, these roots are given by an equation of the form

$$w^m + A_1 w^{m-1} + \cdots + A_m = 0,$$

where $A_j(z_1, \cdots, z_n)$, $j = 1, \cdots, m$, *is analytic in* S *and vanishes at* $(a_1, \cdots, a_n)$.

This theorem is due to Weierstrass, who deduced it from his theorem of factorization, to which we turn in § 3. But the proof is essentially simpler than the one he gave and rests, as the reader perceives, on material already furnished by Cauchy in that remarkable memoir which contained the proof of Taylor's theorem for analytic functions of a complex variable.

This theorem is used in practice much oftener than the general theorem of factorization, and it is well to have a simple proof of it.

§ 3. Weierstrass's Theorem of Factorization.

Let

$$f(w, z) = w^m + A_1 w^{m-1} + \cdots + A_m,$$

and form the function

$$\frac{F(w, z)}{f(w, z)}.$$

Assign to z an arbitrary value in S and hold it fast. Then the above expression becomes a function of w alone:

$$\frac{F(w, z)}{f(w, z)} = \frac{F(w, z)}{(w - w_1) \cdots (w - w_m)} = \Phi(w).$$

This function is analytic within K except for removable singularities in each of the points w_k, the numerator and the denominator having roots of the same order there. If it is defined in these points as equal to its limiting value, it then becomes a function analytic without exception throughout the interior of K and nowhere zero there.

Moreover, $\Phi(w)$ is continuous on the boundary of K. Hence it can be represented by Cauchy's integral formula:

$$\Phi(w) = \frac{1}{2\pi i} \int_C \frac{F(W, z)}{f(W, z)} \cdot \frac{dW}{W - w}.$$

The expression on the right of this equation represents, however, a function $\Omega(w, z)$ of the two independent variables w, z analytic throughout the cylindrical region

$$|w| < r, \qquad |z - a| < h;$$

and we have already observed that it is never zero there.

We have, then, the relation

$$F(w, z) = f(w, z)\Omega(w, z),$$

and this equation is Weierstrass's Theorem of Factorization.

The above analysis holds with only formal modifications for the case of a function $F(w, z_1, \cdots, z_n)$, and hence we may state the result as follows.

WEIERSTRASS'S THEOREM OF FACTORIZATION. *Let $F(w, z_1, \cdots, z_n)$ be analytic in the point $(b, a_1, \cdots, a_n)$, and let*

$$F(b, a_1, \cdots, a_n) = 0, \qquad F(w, a_1, \cdots, a_n) \not\equiv 0.$$

For simplicity, let $b = 0$.

Then F can be represented throughout a certain neighborhood of the point $(0, a_1, \cdots, a_n)$,

$\mathfrak{T}$: $\qquad |w| < r; \qquad |z_k - a_k| < h_k, \qquad k = 1, \cdots, n,$

in the form

$$F(w, z_1, \cdots, z_n) = (w^m + A_1 w^{m-1} + \cdots + A_m)\Omega(w, z_1, \cdots, z_n),$$

where $A_k = A_k(z_1, \cdots, z_n)$, $k = 1, \cdots, m$, denotes a function analytic in the region

S: $\qquad |z_k - a_k| < h_k, \qquad k = 1, \cdots, n,$

and vanishing in the point $(a_1, \cdots, a_n)$; and Ω is analytic in $\mathfrak{T}$ and does not vanish there.

Weierstrass tells us that he gave this theorem repeatedly in his university lectures at Berlin from 1860 on. He published it in lithographed form in the year 1879, and it was printed in the Funktionenlehre of 1886, p. 107; cf. Werke, 2, p. 135. Poincaré gave a proof of the theorem in his Thèse of 1879.

A number of new proofs of the theorem have been given in recent years. Cf. Bliss, Princeton Colloquium, p. 49, and the Transactions of the Amer. Math. Soc., 13 (1912), p. 133, where further literature is cited.

The Case of Two Variables. In the case $n = 1$ a more general theorem can be stated.

THEOREM. *Let $F(w, z)$ be analytic in the point (b, a), and let*

$$F(b, a) = 0, \qquad F(w, z) \not\equiv 0.$$

For simplicity let $b = 0$.

Then $F(w, z)$ can be represented in the neighborhood of the point $(0, a)$ in the form

$$F(w, z) = z^\lambda f(w, z)\Omega(w, z),$$

where $f(w, z) \equiv 1$ *or else*

$$f(w, z) = w^m + A_1 w^{m-1} + \cdots + A_m,$$

$A_k = A_k(z)$ *being analytic in the point* a *and vanishing there; and* $\Omega(w, z)$ *being analytic in the point* $(0, a)$ *and not vanishing there.*

This theorem was used by Black* and it forms an essential element in the study of envelopes by Risley and MacDonald.† It does not admit the generalization for $n > 1$ which suggests itself.‡

Linear Transformation. Let $\Phi(x_0, x_1, \cdots, x_n)$ be a function of $n + 1$ complex variables analytic in the point $(a_0, a_1, \cdots, a_n)$ and vanishing there, but not vanishing identically. It may happen that

$$\Phi(x_0, a_1, \cdots, a_n) \equiv 0,$$

and that the same is true if any other x_k be singled out. It is then impossible to apply the theorem of factorization to the function Φ as it stands.

By means, however, of a suitable linear transformation,

$$\begin{aligned} w &= c_{00}x_0 + c_{01}x_1 + \cdots + c_{0n}x_n + c_0, \\ z_1 &= c_{10}x_0 + c_{11}x_1 + \cdots + c_{1n}x_n + c_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_n &= c_{n0}x_0 + c_{n1}x_1 + \cdots + c_{nn}x_n + c_n, \end{aligned}$$

it is possible to carry Φ over into a function F:

$$\Phi(x_0, x_1, \cdots, x_n) = F(w, z_1, \cdots, z_n)$$

which does satisfy all the conditions of the theorem of factorization, and hence to factor the latter function.§

§ 4. Factors im Kleinen.

Division. Let $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ both be analytic at the point $(a_1, \cdots, a_n)$ and let

$$\Phi(z_1, \cdots, z_n) \not\equiv 0.$$

If there exists a function $Q(z_1, \cdots, z_n)$, also analytic at this point, for which the relation

$$F(z_1, \cdots, z_n) = Q(z_1, \cdots, z_n)\Phi(z_1, \cdots, z_n)$$

* Proceedings Amer. Acad. Arts and Sci., vol. 37 (1902), p. 325.

† Annals of Math., (2) 12 (1911), p. 73.

‡ The Madison Colloquium, Lecture 3, § 2, and Trans. Am. Math. Soc., 17 (1916), p. 1.

§ Weierstrass, l. c.

holds throughout the neighborhood of the point $(a_1, \cdots, a_n)$, then F is said to be *divisible by* Φ *at* or *in the point* $(a_1, \cdots, a_n)$. The function $Q(z_1, \cdots, z_n)$ is called the *quotient* of F by Φ in $(a_1, \cdots, a_n)$, and Φ is called a *factor* of F in or at (a). At those points of the neighborhood of $(a_1, \cdots, a_n)$ in which Φ does not vanish, Q is the arithmetic quotient of F by Φ. But Q is also defined in the points, if any are present, in which $\Phi = 0$, by the method of continuity.

It is to be noted that division by a function which vanishes identically is excluded by the definition.

If F is divisible by Φ at a point (a), then F is divisible by Φ at every other point of a certain neighborhood of (a).

Reducible and Irreducible Functions; Prime Factors. If $F(z_1, \cdots, z_n)$ can be written, in the neighborhood of the point (a), in the form:

$$F(z_1, \cdots, z_n) = F_1(z_1, \cdots, z_n)F_2(z_1, \cdots, z_n),$$

where F_1 and F_2 are both analytic in (a) and vanish there, but neither vanishes identically, then F is said to be *reducible in* or *at the point* (a).

If, on the other hand, $F(z_1, \cdots, z_n)$ is analytic at the point (a) and vanishes there, without however vanishing identically, and if no factorization of the above form is possible, then F is said to be *irreducible in* or *at the point* (a).

The above definitions are seen to relate only to functions $F(z_1, \cdots, z_n)$ which are analytic in a point (a) and vanish there, but do not vanish identically. Other functions $F(z_1, \cdots, z_n)$, which are analytic in the point (a), namely, those which do not vanish there, and the constant 0, are neither reducible nor irreducible in (a).

It does not follow, because a function is irreducible at a point (a), that it is also irreducible at others of its roots which lie near (a). Thus

$$F(x, y, z) = z^2 - x^2y$$

is readily shown to be irreducible at the origin, $(x, y, z) = (0, 0, 0)$. But it is reducible at every root $(0, h, 0)$ for which $h \neq 0$; for

$$F(x, y, z) = (z - x\sqrt{y})(z + x\sqrt{y}).$$

On the other hand, this function is irreducible at the points $(a, 0, 0)$ for all values of a.

An irreducible factor of a function at a given point is called a *prime factor* of the function *at that point.*

Equivalent Functions. Two functions, $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$, are said to be *equivalent* to each other *in* or *at the point* (a) if each is divisible by the other there.

In order that F be equivalent to Φ in (a) it is necessary and sufficient that F be divisible by Φ at (a):

$$F(z_1, \cdots, z_n) = Q(z_1, \cdots, z_n)\Phi(z_1, \cdots, z_n),$$

and furthermore that Q shall not vanish at (a).

If two functions are equivalent to each other at a point, they are also equivalent at every other point of a certain neighborhood of the given point.

We turn next to the statement of one of the most important theorems with which this paper deals.

FUNDAMENTAL THEOREM. *If $F(z_1, \cdots, z_n)$ is analytic in the point $(a_1, \cdots, a_n)$ and vanishes there, but does not vanish identically, then F can be represented in one, and essentially in only one, way as the product of prime factors.*

By *essentially* is meant that, if F be represented by a second product of prime factors, each prime factor of the second representation will be equivalent to a prime factor of the first representation, and the multiplicities of corresponding prime factors in the two representations will be respectively equal.

For the case $n = 1$ the proof of the theorem is immediate. The proof for $n > 1$ is given in § 8.

If the functions F and Φ are analytic in the point (a) and vanish there, and if they have no common divisor which vanishes there, they are said to be *relatively prime in the point* (a), or to have *no common divisor in* (a). And likewise, if F and Φ are both divisible in (a) by a function Ψ which vanishes there, they are said to have a *common divisor in* (a), namely Ψ.

If two functions are relatively prime in a point (a), they are also relatively prime in every other common root of a certain neighborhood of (a).*

The proof of this theorem will be given at the close of § 9.

Greatest Common Divisor. Let $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ be two functions which are analytic in a point (a), and let neither vanish identically. The product of all the irreducible common factors of F and Φ in (a), each taken with the lower of the two multiplicities with which it occurs in F and Φ, is called the *greatest common divisor of F and Φ in (a)*.

The roots of this latter function in the neighborhood of (a) yield all the points of that neighborhood in which F and Φ have a common factor, and this function is also the greatest common divisor of F and Φ in each of these points. The proof of this theorem follows at once from the theorem of Weierstrass's just cited.

* Weierstrass, Werke, 2, p. 154.

The least common multiple of F and Φ in a point (a) is defined in a similar manner.

Invariance of the Foregoing Definitions and Theorems. It is obvious that, if the variables $z_1, \cdots, z_n$ be subjected to a non-singular linear transformation,

$$z_i' = c_{i1}z_1 + \cdots + c_{in}z_n + c_i, \qquad i = 1, \cdots, n,$$

a function $F(z_1, \cdots, z_n)$ or a pair of functions F and Φ which satisfy a given one of the above definitions or theorems in a point (a) will go over into a function $F'(z_1', \cdots, z_n')$ or a pair of functions F', Φ' satisfying the same condition in (a').

More generally, the same remark is true if instead of a linear transformation, an arbitrary non-singular analytic transformation is made:

$$z_i' = \varphi_i(z_1, \cdots, z_n), \qquad i = 1, \cdots, n,$$

where, then, each of the functions φ_i is analytic in the point (a) and their Jacobian,

$$\frac{\partial(\varphi_1, \cdots, \varphi_n)}{\partial(z_1, \cdots, z_n)}$$

does not vanish there.

The proof of the fundamental theorem rests on the theory of algebroid polynomials—a class of functions also having independent importance.

§ 5. Algebroid Polynomials.*

By an *algebroid polynomial* is meant a function

$$(A) \qquad f(x, y) = f(x, y_1, \cdots, y_n) = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$

where $a_k = a_k(y_1, \cdots, y_n)$, $k = 0, \cdots, m$, is analytic in the point $(y) = (c)$. For simplicity, we shall usually take the point (c) at the origin, $(c) = (0)$.

If all the coefficients a vanish identically, then f vanishes identically: and conversely. If f does not vanish identically, let

$$a_0(y_1, \cdots, y_n) \not\equiv 0.$$

Then m is defined as the *degree* of f. An algebroid polynomial of degree 0 is any function of $(y_1, \cdots, y_n)$ which is analytic in the origin and does not vanish identically.

We know the fundamental theorem of § 4 to hold for one variable,

* The theory here set forth is closely allied to the theory of ordinary polynomials, and a knowledge of this theory, as given for example in Bôcher's Algebra, Ch. 16, is presupposed on the part of the reader. For the latter's convenience we have changed the notation to conform in the main with that of the reference.

and we assume it to hold for 2, $\cdots$, n variables. By means of the theory developed in this paragraph we shall be able to show that it holds for $n + 1$ variables, and thus establish it generally.

LEMMA 1. *If the algebroid polynomial* (A) *is divisible at the origin by a function* $\psi(y) = \psi(y_1, \cdots, y_n)$, *then every coefficient of* f *is divisible by* $\psi(y)$ *in the origin.*

Since division is subject to the definition of § 4, we have by hypothesis

$$f(x, y) = \omega(x, y)\psi(y), \tag{1}$$

where ω is analytic in the $n + 1$ variables $(x, y_1, \cdots, y_n)$ at the origin, but not by hypothesis is ω an algebroid polynomial. On developing ω according to powers of x:

$$\omega(x, y) = c_0 + c_1x + c_2x^2 + \cdots,$$

substituting this series in the right-hand member of (1), and comparing coefficients, the truth of the lemma becomes apparent.

An algebroid polynomial $f(x, y)$ is said to be *primitive* if it is not divisible by any function $\psi(y)$ analytic in the origin and vanishing there. If in particular a primitive algebroid polynomial is of degree 0, it cannot vanish in the origin; and conversely, any function $\psi(y)$ analytic in the origin and not vanishing there is a primitive algebroid polynomial of degree 0.

COROLLARY. *If* $f(x, y)$ *is an algebroid polynomial which does not vanish identically, it can be written as the product of two algebroid polynomials,*

$$f(x, y) = \psi(y)g(x, y),$$

of which the first is of degree 0 *and the second is primitive.*

If, namely, $f(x, y)$ is itself primitive, then $\psi(y)$ can be set $= 1$. If not, $\psi(y)$ can be taken as the greatest common divisor of the coefficients of f.

LEMMA 2. *If the product of two algebroid polynomials,* $f(x, y)$ *and* $\varphi(x, y)$, *is divisible in the origin by a function* $\psi(y)$ *irreducible in the origin, then one of the two factors is divisible by* $\psi(y)$ *in the origin.*

The proof is the same as in the case of ordinary polynomials; cf. Bôcher, l. c., § 73, Theorem 2.

Division. Let $f(x, y)$ and $\varphi(x, y)$ be two algebroid polynomials, of which the second does not vanish identically:

$$\varphi(x, y) = b_0x^p + b_1x^{p-1} + \cdots + b_p,$$
$$b_0(y_1, \cdots, y_n) \not\equiv 0, \qquad 0 \leqq p.$$

Then there exists between f and φ a relation of the following form:

$$P(y)f(x, y) = Q(x, y)\varphi(x, y) + R(x, y), \tag{2}$$

where $P(y)$, $Q(x, y)$, $R(x, y)$ are algebroid polynomials, of which the first is of degree 0 and the last, $R(x, y)$, either vanishes identically or is of lower degree than $\varphi(x, y)$. Moreover, P, Q, R shall admit no common factor in the origin.

The proof is the same as for polynomials; Bôcher, l. c., § 63. The relation (2) was given by Weierstrass; cf. Werke, 2, p. 144.

Let

$$P_1(y)f(x, y) = Q_1(x, y)\varphi(x, y) + R_1(x, y)$$

be a second relation satisfying the same conditions. Then $P(y)$ and $P_1(y)$ are equivalent to each other in the origin. Moreover, $R(x, y)$ and $R_1(x, y)$ either both vanish identically or else are equivalent to each other in the origin; and the same is true of $Q(x, y)$ and $Q_1(x, y)$.

To prove this, consider the relation obtained at once from the two given equations:

$$0 = [P_1(y)Q(x, y) - P(y)Q_1(x, y)]\varphi(x, y) + [P_1(y)R(x, y) - P(y)R_1(x, y)].$$

The second [ ] vanishes identically; for otherwise, if $\varphi(x, y)$ be of positive degree, it would be of lower degree than $\varphi(x, y)$, and would be divisible by $\varphi(x, y)$. If φ is of degree 0, then both R and R_1 vanish identically.

It follows, then, that the first [ ] must also vanish identically.

$P(y)$ and $P_1(y)$ are now shown to be equivalent in the origin. For otherwise one of these functions, as $P(y)$, must admit a factor $\psi(y)$ irreducible in the origin which does not divide the other function. From the vanishing of both brackets we infer that $\psi(y)$ divides both $Q(x, y)$ and $R(x, y)$, and this is contrary to hypothesis. The remainder of the proof is now obvious.

Algebraic and Analytic Division. From the analogy with ordinary polynomials it is natural to define f as divisible by φ when, in the relation (2), $R(x, y)$ vanishes identically and $P(y)$ does not vanish, so that it can be set equal to unity:

$$f(x, y) = Q(x, y)\varphi(x, y), \tag{3}$$

where $Q(x, y)$ is an algebroid polynomial. When this condition is fulfilled, we shall say that f is *algebraically divisible by φ in the point* $(y) = (0)$, or more generally, $(y) = (c)$.

On the other hand, we already have a definition of divisibility in a point, according to which a relation of the same form, (3), must hold; but here $Q(x, y)$ need not be an algebroid polynomial—it is sufficient that Q be analytic in the point $(x, y_1, \cdots, y_n) = (0, 0, \cdots, 0)$ or $(c, c_1, \cdots, c_n)$.

In distinction from algebraic divisibility just defined we will describe the earlier case now by saying that $f(x, y)$ is *analytically divisible* by $\varphi(x, y)$ in the point $(x, y_1, \cdots, y_n) = (c, c_1, \cdots, c_n)$.

When f is algebraically divisible by φ in the point $(y) = (c)$, it is obviously analytically divisible by φ in any point

$$(x, y_1, \cdots, y_n) = (c, c_1, \cdots, c_n), \qquad |c| < \infty.$$

But if f is analytically divisible by φ in a point $(c, c_1, \cdots, c_n)$, it does not follow that f is algebraically divisible by φ in the point $(y) = (c)$. For example, let

$$f(x, y) = x + y,$$

$$\varphi(x, y) = x^2 + (y + 1)x + y = (x + y)(x + 1).$$

Then f is analytically divisible by φ in the origin,

$$Q(x, y) = \frac{1}{x + 1} = 1 - x + x^2 - \cdots.$$

But f is not algebraically divisible by φ in the point $y = 0$.

We can, however, state the following criterion for the coincidence of the two definitions.

THEOREM 1. *In order that the algebroid polynomial $f(x, y)$ be algebraically divisible by the algebroid polynomial $\varphi(x, y)$ in the point*

$$(y_1, \cdots, y_n) = (c_1, \cdots, c_n)$$

it is necessary and sufficient that $f(x, y)$ be analytically divisible by $\varphi(x, y)$ in every point of a certain region

$$\mathfrak{T}: \qquad |x| < \infty, \qquad |y_k - c_k| < h, \qquad k = 1, \cdots, n,$$

*where h is a positive constant, no matter how small.**

The condition is obviously necessary. To prove it sufficient, form the function

$$\frac{f(x, y)}{\varphi(x, y)} = \Omega(x, y).$$

Then Ω has only removable singularities in $\mathfrak{T}$, and shall be defined in these points by its limiting value there. Hence Ω can be developed into a series of the form

$$\Omega(x, y) = c_0 + c_1 x + c_2 x^2 + \cdots,$$

* It would not be sufficient if f were analytically divisible by φ merely in the points

$$(x, y_1, \cdots, y_n) = (c, 0, \cdots, 0),$$

where c is arbitrary, as the example

$$f(x, y) = x, \qquad \varphi(x, y) = yx^2 - x$$

shows.

where each c_k is a function of (y) analytic throughout the region

$$S: \qquad |y_k| < h, \qquad k = 1, \cdots, n,$$

the series converging for all values of x.

Let (y') be a point of S in which the coefficient of the highest power of x in φ is not 0:

$$b_0(y_1', \cdots, y_n') \neq 0,$$

and let σ be a neighborhood of (y') lying wholly within S and such that

$$\eta < |b_0(y_1, \cdots, y_n)|$$

for every point (y) of σ, η being a suitable positive constant. Then, when (y) lies in σ, the roots of $\varphi(x, y)$ remain finite. Hence, if $m \geqq p$, the function

$$\frac{f(x, y)}{x^{m-p}\varphi(x, y)} = \frac{c_0}{x^{m-p}} + \cdots + c_{m-p} + c_{m-p+1}x + \cdots$$

remains finite when (y) lies in σ and $|x| > G$, where G denotes a suitable positive number. It follows, then, that

$$c_k(y_1, \cdots, y_n) \equiv 0, \qquad m - p < k.$$

If $m < p$, $\Omega(x, y)$ would have to vanish identically, and thus this case is impossible. Finally, if $f(x, y)$ vanishes identically, the theorem is granted.

Algebroid Polynomials of Class A. An important class of algebroid polynomials

$$f(x, y) = a_0(x - c)^m + a_1(x - c)^{m-1} + \cdots + a_m,$$

where $a_k = a_k(y_1, \cdots, y_n)$ is analytic in the point $(c_1, \cdots, c_n)$, consists of those for which $m > 0$ and

$$a_0(c_1, \cdots, c_n) \neq 0, \qquad a_k(c_1, \cdots, c_n) = 0, \qquad k = 1, \cdots, m.$$

These shall be denoted as *algebroid polynomials of Class A*. The point $(c, c_1, \cdots, c_n)$ shall be called the *vertex* of f.

An algebroid polynomial $f(x, y)$ of Class A is necessarily primitive. Furthermore, to an arbitrarily small positive ϵ corresponds a positive δ such that, if (y) lies in the region

$$|y_k - c_k| < \delta, \qquad k = 1, \cdots, n,$$

then each root of $f(x, y)$ lies in the circle

$$|x - c| < \epsilon.$$

THEOREM 2. *If an algebroid polynomial $f(x, y)$ is divisible analytically by an algebroid polynomial $\varphi(x, y)$ of Class A in the vertex $(c, c_1, \cdots, c_n)$ of the latter, then $f(x, y)$ is algebraically divisible by $\varphi(x, y)$ in the point $(y) = (c)$.*

For, all the hypotheses of Theorem 1 are fulfilled throughout a definite region

$$|x| < \infty, \qquad |y_k - c_k| < \delta, \qquad k = 1, \cdots, n.$$

COROLLARY. *If $f(x, y)$ is also an algebroid polynomial of Class A, the quotient will be an algebroid polynomial of Class A, or else an algebroid polynomial of the 0th degree which does not vanish in the origin.*

In particular, if the coefficient of the highest power of $x - c$ in both f and φ is unity, the coefficient of the highest power of $x - c$ in the quotient will also be unity, or else the quotient will reduce identically to unity.

§ 6. The Algorithm of the Greatest Common Divisor.

Euclid's algorithm of the greatest common divisor, as applied to polynomials in several variables (Bôcher, l. c.), can be used with modifications now pretty obvious for algebroid polynomials. It is understood throughout that all algebroid polynomials are considered in the same point.

Let $f(x, y)$, $\varphi(x, y)$ be two algebroid polynomials of degrees $m \geqq p > 0$ respectively. By successive applications of the relation (2) of § 5 we find:

$$\text{I.} \quad \begin{cases} P_{-1}(y) f(x, y) = Q_0(x, y)\, \varphi(x, y) + R_1(x, y), \\ P_0(y)\, \varphi(x, y) = Q_1(x, y)\, R_1(x, y) + R_2(x, y), \\ \cdots\cdots\cdots\

$\varphi(x, y)$, each of positive degree, may have an algebraic common divisor, it is necessary and sufficient that

$$R_{\rho+1}(y) \equiv 0.$$

The condition is evidently necessary. To show that it is sufficient. Here, by hypothesis, it is fulfilled. Let

$$R_\rho(x, y) = \psi(y)G(x, y),$$

where $\psi(y)$ and $G(x, y)$ are algebroid polynomials of the 0th and positive degrees respectively, and $G(x, y)$, moreover, is primitive. Then G is shown at once by the algorithm to divide both f and φ.

THEOREM 2. *If $R_{\rho+1}(y) \equiv 0$, and if $G(x, y)$ has the same meaning as above, then $G(x, y)$ is a greatest algebraic common divisor of f and φ.*

Any two greatest algebraic common divisors differ by a factor which is an algebroid polynomial of degree 0.

The proof is given at once by the algorithm, as in the case of polynomials. Every primitive algebraic common divisor of f and φ divides R_ρ and hence G.

THEOREM 3. *If two algebroid polynomials $f(x, y)$ and $\varphi(x, y)$, each of positive degree, have no algebraic common factor, then three algebroid polynomials $A(x, y)$, $B(x, y)$, $R(y)$, no one of which vanishes identically, can be found such that*

II. $$A(x, y)f(x, y) + B(x, y)\varphi(x, y) = R(y).$$

And conversely, if such a relation exists, f and φ have no algebraic common factor.

§ 7. Analogue of the Fundamental Theorem for Algebroid Polynomials.

An algebroid polynomial $f(x, y)$ shall be said to be *algebraically reducible* if it can be written as the product of two algebroid polynomials each of positive degree. If it cannot be so written, and if it is of positive degree it shall be said to be *algebraically irreducible*. Thus the algebroid polynomial

$$f(x, y) = yx$$

is algebraically irreducible.

An algebroid polynomial of the 0th degree, and the polynomial 0, are neither reducible nor irreducible.

If an algebroid polynomial of Class A is algebraically reducible, its factors are both algebroid polynomials of Class A.

The analogue of the fundamental theorem of § 4 for algebroid polynomials is as follows.

THEOREM 1. *An algebroid polynomial of positive degree, $f(x, y)$, can be*

written in one, and essentially in only one, way as the product of algebraically irreducible factors:

$$f(x, y) = [f_1(x, y)]^{\lambda_1} \cdots [f_l(x, y)]^{\lambda_l}.$$

The proof of this theorem rests on the following lemma.

LEMMA. *Let $f(x, y)$, $g(x, y)$ and $\varphi(x, y)$ be three algebroid polynomials, of which the last is primitive and algebraically irreducible; and let $f(x, y)$ not be algebraically divisible by $\varphi(x, y)$. If the product $f(x, y)\, g(x, y)$ is algebraically divisible by $\varphi(x, y)$, then $g(x, y)$ is algebraically divisible by $\varphi(x, y)$.*

The proof of the lemma is given as in the case of ordinary polynomials, either Euclid's algorithm or the relation II. being used for this purpose. Hence the above theorem is established.

Algebroid Polynomials of Class A. For these the two definitions of reducibility—algebraic reducibility and the analytic reducibility defined in § 4—come together. In other words:

THEOREM 2. *In order that an algebroid polynomial of Class A be algebraically irreducible in the point $(y) = (c)$, it is necessary and sufficient that it be analytically irreducible in its vertex $(x, y_1, \cdots, y_n) = (c, c_1, \cdots, c_n)$.*

The condition is obviously sufficient. To show that it is also necessary, let $f(x, y)$ be algebraically irreducible, and suppose

$$f(x, y) = \varphi(x, y)\psi(x, y), \tag{4}$$

where $\varphi(x, y)$ and $\psi(x, y)$ are both analytic in the vertex $(c, c_1, \cdots, c_n)$ of f and φ, and vanish there. Then

$$\varphi(x, 0) \not\equiv 0,$$

since otherwise $f(x, 0)$ would vanish identically. Hence, by Weierstrass's Theorem of Factorization $\varphi(x, y)$ can be taken as an algebroid polynomial of Class A.

From Theorem 2 of § 5 and its corollary it now follows that the other factor, $\psi(x, y)$, is also an algebroid polynomial of Class A, or else an algebroid polynomial of the 0th degree, which does not vanish in the origin. Both of these cases lead to a contradiction, and thus the theorem is established.

§ 8. Proof of the Fundamental Theorem in the General Case.

We turn now to the proof of the fundamental theorem of § 4 in the general case. Let a linear transformation be made whereby the variables $(z_1, \cdots, z_n)$ are replaced by the variables $(x, y_1, \cdots, y_\nu)$, $\nu = n - 1$, and $F(z_1, \cdots, z_n)$ goes over into a function $\mathfrak{F}(x, y_1, \cdots, y_\nu)$ which can be

treated by the theorem of factorization in its original form:

$$(z) \sim (x, y) = (0, 0);$$

$$\mathfrak{F}(x, 0, \cdots, 0) \not\equiv 0,$$

$$F(z_1, \cdots, z_n) = \mathfrak{F}(x, y_1, \cdots, y_\nu) = f(x, y)\Omega(x, y),$$

where $f(x, y)$ is an algebroid polynomial of Class A with its vertex in the origin.

By the theorem of the last paragraph $f(x, y)$ can be written as the product of a finite number of factors each analytically irreducible in the origin; and since such factors remain irreducible when a linear transformation is made, one half of the proof is thus given, namely, that $F(z_1, \cdots, z_n)$ can be written at least in one way as the product of a finite number of factors, each irreducible in the point $(z) = (a)$:

$$F(z_1, \cdots, z_n) = [F_1(z_1, \cdots, z_n)]^{\lambda_1} \cdots [F_l(z_1, \cdots, z_n)]^{\lambda_l}.$$

Suppose a second factorization of this sort were possible:

$$F(z_1, \cdots, z_n) = [\Phi_1(z_1, \cdots, z_n)]^{\mu_1} \cdots [\Phi_m(z_1, \cdots, z_n)]^{\mu_m}.$$

Then the same linear transformation as was used before will carry each function $\Phi_k(z_1, \cdots, z_n)$ into a function

$$\Psi_k(x, y_1, \cdots, y_\nu), \qquad \Psi_k(x, 0, \cdots, 0) \not\equiv 0,$$

to which Weierstrass's Theorem of Factorization is applicable. Hence

$$\begin{aligned} \mathfrak{F}(x, y) &= f(x, y)\Omega(x, y) \\ &= [f_1(x, y)]^{\lambda_1} \cdots [f_l(x, y)]^{\lambda_l}\Omega(x, y) \\ &= [\varphi_1(x, y)]^{\mu_1} \cdots [\varphi_q(x, y)]^{\mu_q} \mathbf{X}(x, y) \end{aligned}$$

and

$$f(x, y) = [\varphi_1(x, y)]^{\mu_1} \cdots [\varphi_q(x, y)]^{\mu_q} \frac{\mathbf{X}(x, y)}{\Omega(x, y)}.$$

Thus the algebroid polynomial of Class A, $f(x, y)$, is seen to be divisible analytically at its vertex by an algebroid polynomial of Class A. By Theorem 2 of § 5 and its corollary the quotient is also an algebroid polynomial, and since it does not vanish in the origin, it does not depend on x.

Finally, then, by the theorems of § 7 the factors $f_k(x, y)$ and $\varphi_k(x, y)$ are respectively equivalent to each other and occur with like multiplicities. This proves the theorem.

§ 9. A General Theorem Relating to Divisibility.

A theorem of frequent application in algebraic geometry is the following. If two algebraic curves have ever so restricted an arc in common, they have a whole irreducible algebraic curve in common.

The theorem can be generalized for arbitrary analytic hypersurfaces as follows.

THEOREM.* *If* $F(z_1, \cdots, z_n)$ *and* $\Phi(z_1, \cdots, z_n)$ *are two functions each analytic in the point* $(z) = (a)$ *and vanishing there, neither function vanishing identically, and if* F *vanishes for every root of* Φ *which lies in the neighborhood of* (a), *then each irreducible factor of* Φ *in* (a) *is also a factor of* F *in* (a).

It is evidently sufficient to prove the theorem for the case that Φ is irreducible in (a). Let a linear transformation be made whereby F and Φ go over into functions to which the theorem of factorization in its first form is applicable:

$$F(z_1, \cdots, z_n) = \mathfrak{F}(x, y_1, \cdots, y_{n-1}), \qquad \Phi(z_1, \cdots, z_n) = \Psi(x, y_1, \cdots, y_{n-1}),$$

$$\mathfrak{F}(x, 0, \cdots, 0) \not\equiv 0, \qquad \Psi(x, 0, \cdots, 0) \not\equiv 0,$$

$$\mathfrak{F}(x, y) = f(x, y)\Omega(x, y); \qquad \Psi(x, y) = \psi(x, y)\mathrm{X}(x, y).$$

Here, $\psi(x, y)$ is an irreducible algebroid polynomial of Class A. If the algebroid polynomial $f(x, y)$ does not admit $\psi(x, y)$ as a factor, then f is algebraically prime to ψ, and by II, § 6, we have:

$$A(x, y)f(x, y) + B(x, y)\psi(x, y) = R(y).$$

Let (y') be a point in the neighborhood of the origin, for which

$$R(y') \neq 0,$$

and let x' be a root of $\psi(x, y')$. Then the left-hand side of this equation vanishes at the point (x', y'), while the right-hand side does not. From this contradiction follows the truth of the theorem.

Proof of the Last Theorem of § 4. Let $F(z_1, \cdots, z_r)$ and $\Phi(z_1, \cdots, z_n)$ be relatively prime in the point $(z) = (a)$, and let them be transformed as above into functions $\mathfrak{F}(x, y)$ and $\Psi(x, y)$. Then there exists a relation of the form

$$A(x, y)f(x, y) + B(x, y)\psi(x, y) = R(y),$$

where $R(y)$ does not vanish identically. If, now, (z^0) be any point of the neighborhood of (a) and (x^0, y^0) the corresponding point, this same relation holds in the neighborhood of the latter point, and hence f and ψ are relatively prime at (x^0, y^0). This shows that F and Φ are likewise relatively prime at (z^0), *q.e.d.*

HARVARD UNIVERSITY,
May, 1917.

* Hölder, Mathematisch-naturwissenschaftliche Mitteilungen, I (1884); Tübingen, Fues. Study, Ternäre Formen, p. 202.

AN APPLICATION OF FOURIER'S SERIES TO PROBABILITY.

By David F. Barrow.

Introduction.

Many persons have been interested in compounding frequencies. They all seem to aim at asymptotic expressions which represent approximately the functions obtained by compounding a large number of frequencies. The most general theorem of this sort is to the effect that if a great many frequencies be compounded the result approaches a Gaussian error function.* The generality and beauty of this theorem, and the fact that in most applications a great many frequencies enter, seem to have led writers to give little attention to those cases where only a few frequencies enter. Furthermore in those cases where exact formulas for compounding a few frequencies have been given, great difficulty is found in obtaining numerical results. Even Laplace confesses that in one case such a difficulty arrested him for a long time before he found a suitable approximation.

This article gives a simple method of compounding frequency functions when they can be represented by Fourier's series. It has the advantages of being applicable over a wide range of problems and of yielding formulas which represent exactly the frequencies and from which numerical results can be obtained with relative ease. Owing to the existence of asymptotic formulas, it is most useful where a few frequencies are to be compounded, that is where the asymptotic solution is not very exact.

In section 1 the general method is set forth. In section 2 a particular problem is solved. The solution looks so unlike the asymptotic solution that it seems of interest to compare the two directly to get an upper limit on the discrepancy. This is done in sections 4 and 5, and also a way is pointed out for comparing the exact and asymptotic solutions in various other cases.

1. Fundamental theorems.

In many probability problems we consider a variable t which may take on any one of an infinite number of values, for instance any value in some interval, finite or infinite. Of course in such a case the probability

* For an exact statement and a simple proof of this theorem see Crofton, Phil. Trans., vol. 160, p. 175.

that any particular value will be assumed is infinitesimal, but it is often possible to define a function $f(x)$, called the frequency of t, such that the integral of $f(x)$ from any value a to any larger value b gives the probability that t will assume a value in the interval (a, b). Evidently $f(x)$ must be everywhere positive or zero, and the integral of $f(x)$ from $-\infty$ to $+\infty$ must be unity since this is the symbol for certainty.

LEMMA. *If $f(x)$ is the frequency of t, and $\varphi(x)$ is the frequency of τ, and $F(x)$ is the frequency of $t + \tau$; then*

$$F(x) = \int_{-\infty}^{\infty} f(x - \lambda)\varphi(\lambda)d\lambda = \int_{-\infty}^{\infty} f(\mu)\varphi(x - \mu)d\mu.$$

Proof. The probability that $t + \tau$ will fall between x and $x + \Delta x$ may be expressed in two ways. First it is by definition the integral of $F(\lambda)$ from x to $x + \Delta x$, and secondly it is the double integral of $f(\mu)\varphi(\lambda)$ extended over that region of the λ, μ-plane which lies between the two lines $\lambda + \mu = x$ and $\lambda + \mu = x + \Delta x$. Equating these two expressions we have

$$\int_x^{x+\Delta x} F(\lambda)d\lambda = \int_{-\infty}^{\infty}\int_{x-\lambda}^{x+\Delta x-\lambda} \varphi(\lambda)f(\mu)d\mu d\lambda = \int_{-\infty}^{\infty}\int_{x-\mu}^{x+\Delta x-\mu} \varphi(\lambda)f(\mu)d\lambda d\mu.$$

Dividing by Δx and taking the limit for $\Delta x = 0$ we obtain the desired result.

THEOREM 1. *If*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}\left(a_i \cos\frac{i\pi x}{C} + b_i \sin\frac{i\pi x}{C}\right)$$

and

$$\varphi(x) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty}\left(\alpha_i \cos\frac{i\pi x}{C} + \beta_i \sin\frac{i\pi x}{C}\right)$$

and

$$F(x) = \tfrac{1}{2}A_0 + \sum_{i=1}^{\infty}\left(A_i \cos\frac{i\pi x}{C} + B_i \sin\frac{i\pi x}{C}\right)$$

represent the frequencies of t, τ, and $t + \tau$ respectively in the interval $(-C, C)$, while outside this interval all three functions are identically zero; then

$$A_0 = Ca_0\alpha_0,$$

$$A_i = C(a_i\alpha_i - b_i\beta_i),$$

$$B_i = C(a_i\beta_i + b_i\alpha_i) \qquad (i = 1, 2, 3, \cdots).$$

Proof. In applying the lemma to calculate $F(x)$ from $f(x)$ and $\varphi(x)$, the limits of integration may be reduced to $-C$ and C, since outside this

interval the integrand is identically zero. Now

$$f(x-\lambda) = \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}\left[a_i \cos\frac{i\pi}{C}(x-\lambda) + b_i \sin\frac{i\pi}{C}(x-\lambda)\right]$$

$$= \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}\left[a_i \cos\frac{i\pi x}{C}\cos\frac{i\pi\lambda}{C} + a_i \sin\frac{i\pi x}{C}\sin\frac{i\pi\lambda}{C} + b_i \sin\frac{i\pi x}{C}\cos\frac{i\pi\lambda}{C} - b_i \cos\frac{i\pi x}{C}\sin\frac{i\pi\lambda}{C}\right].$$

To obtain $F(x)$ we multiply this series, term by term, by the series representing $\varphi(\lambda)$, and integrate with respect to λ from $-C$ to C.* On account of the orthogonal property of the sines and cosines involved a great many of the resulting terms are zero. The coefficients of the remaining terms give us the above values for A and B in terms of a, b, α and β.

We have slurred over one difficulty, namely that when we represent the function $f(x-\lambda)$ by the Fourier's series where x and λ can each range over the interval $(-C, C)$, the value of the argument $x-\lambda$ may lie outside this interval and for such values the series may fail to represent $f(x-\lambda)$ correctly. We shall show that whenever this occurs the series representing $\varphi(\lambda)$ vanishes so that the result of the integration is not vitiated. Let (d_1, d_2) and (δ_1, δ_2) be the smallest intervals outside of which $f(x)$ and $\varphi(x)$ respectively vanish identically. Then the corresponding interval for $F(x)$ will be $(d_1+\delta_1, d_2+\delta_2)$. Now the interval $(-C, C)$ must by hypothesis include these three intervals. Hence the series for $f(x)$ must converge to zero in the two intervals $(-C, d_1)$ and (d_2, C), and since it is periodic it will converge to zero for a distance beyond the ends of $(-C, C)$ and hence will represent $f(x)$ in the wider interval $(-2C+d_2, 2C+d_1)$. We need, therefore, look at only two cases: $x-\lambda < -2C+d_2$, and $x-\lambda > 2C+d_1$. Take the first case, then

$$x-\lambda < -2C+d_2;$$

but $$-x < C$$

and $$d_2+\delta_2 < C$$

Adding $$\delta_2 - \lambda < 0 \quad \text{or} \quad \lambda > \delta_2.$$

Hence λ must lie between δ_2 and C, and the Fourier's series for $\varphi(\lambda)$ will converge to zero. We find similarly for the second case that λ must lie between $-C$ and δ_1, so that the series for $\varphi(\lambda)$ is zero in this case also.

* The uniform convergence of Fourier's series is sufficient to permit this operation, cf. Picard, Traite d'Analyse, Vol. 1, chapter on *trigonometric series*.

The results of theorem 1 may be put into the following compact form by defining $b_0, = \beta_0 = 0$, and making use of the imaginary unit $\sqrt{-1}$:

$$A_i + B_i\sqrt{-1} = C(a_i + b_i\sqrt{-1})(\alpha_i + \beta_i\sqrt{-1}) \qquad (i = 0, 1, 2, \cdots).$$

Equating real and imaginary parts gives the formulas. From this form it is evident that to find the frequency of the sum of n variables we need only multiply the n parentheses together and prefix the factor C^{n-1}. Then the real and imaginary parts of the product give the coefficients in the Fourier's series for the frequency of the sum.

An alternate form of theorem 1 is

THEOREM 2. *If*

$$f(x) = \tfrac{1}{2}r_0 + \sum_{i=1}^{\infty} r_i \cos\left(\frac{i\pi x}{C} - g_i\right),$$

$$\varphi(x) = \tfrac{1}{2}\rho_0 + \sum_{i=1}^{\infty} \rho_i \cos\left(\frac{i\pi x}{C} - \gamma_i\right),$$

$$F(x) = \tfrac{1}{2}R_0 + \sum_{i=1}^{\infty} R_i \cos\left(\frac{i\pi x}{C} - G_i\right),$$

represent the frequencies of t, τ, and $t + \tau$ respectively in the interval $(-C, C)$ and outside this interval all three functions are zero, then

$$R_i = Cr_i\rho_i \qquad (i = 0, 1, 2, \cdots),$$

$$G_i = g_i + \gamma_i \qquad (i = 1, 2, 3, \cdots).$$

An interesting special case occurs when the three functions are even in addition to satisfying the conditions named in theorem 1, for then each can be expressed in a series of cosines alone, causing obvious simplifications in our formulas.

Not only can theorem 1 be used to find the frequency for the sum of several variables whose frequencies are known, but it may also be used to reverse the process. For example if the frequency for the sum of two variables is given, and that for one of the variables, then the frequency for the other may be found. Again, if an arbitrary frequency function be given, it may be possible in various ways to assign two or more variables with suitable frequencies so that their sum shall have the given frequency. The writer hopes to take up this question later.

2. Solution of a problem.

As an application of the general method let us solve the following problem.

"There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that

any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, $\cdots$?" *

Let $f(x, n)$ denote the frequency of the sum of n such numbers (or variables). Then $f(x, 1)$ is identically zero outside the interval $(-\frac{1}{2}, \frac{1}{2})$, since the variable whose frequency it is cannot occur outside this interval. Furthermore all values between $-\frac{1}{2}$ and $+\frac{1}{2}$ are equally probable, so that the integral of $f(x, 1)$ over any subinterval thereof should be proportional to the length of that subinterval. Hence $f(x, 1)$ must equal a constant in this interval, and this constant must be unity in order that the integral of $f(x, 1)$ from $-\frac{1}{2}$ to $+\frac{1}{2}$ be equal to unity, the symbol for certainty. Thus $f(x, 1)$ is an even function and may be developed into the following series of cosines:

$$f(x, 1) = \frac{1}{2C} + \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\frac{i\pi}{2C}\cos\frac{i\pi x}{C}.$$

The frequency for the sum of n such variables is easily found, by successive applications of Theorem 1, to be

$$(1) \qquad f(x, n) = \frac{1}{2C} + \frac{2^n C^{n-1}}{\pi^n}\sum_{i=1}^{\infty}\frac{1}{i^n}\sin^n\left(\frac{i\pi}{2C}\right)\cos\frac{i\pi x}{C}.$$

Obviously if C be taken equal to $n/2$ it will be large enough to include all points where the function does not vanish identically. It may be taken larger without damaging the formula, in which case the series merely converges to zero in the intervals $(n/2, C)$ and $(-C, -(n/2))$. In fact, on account of the periodicity of the cosines, the series converges to zero for a distance $(C - (n/2))$ beyond the ends of the interval $(-C, C)$, and therefore represents correctly $f(x, n)$ when x lies in the interval $((n/2) - 2C, 2C - (n/2))$. Now this last statement holds true even when C has a value between $n/4$ and $n/2$, but the proof is too detailed to be given here. The fact is only of importance in calculations, as we shall wish to take C as small as possible in order to obtain a rapidly converging series. A check calculation may be made by changing the value of C.

Now let $\psi(s, n)$ denote the function which furnishes the solution of our problem. Then

$$\psi(s, n) = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} f(x, n)dx = \frac{1}{2C} + \frac{2^{n+1}C^n}{\pi^{n+1}}\sum_{i=1}^{\infty}\frac{1}{i^{n+1}}\sin^{n+1}\left(\frac{i\pi}{2C}\right)\cos\frac{i\pi s}{C}.$$

* The problem was stated in this form by Prof. E. W. Brown in The American Math. Monthly, Nov., 1916, p. 341. It is a special case of a problem solved by Laplace in his Théorie Analytique des Probabilités (Paris, 1820), pp. 253–256. He finds a formula unsuitable for calculation, and so resorts to an approximate formula which is practically the asymptotic Gaussian error function which we derive in section 4. See collected works of Laplace, Vol. 12, pp. 301 ff.

The series represents $\psi(s, n)$ if $(n+1)/2 - 2C \leqq s \leqq 2C - (n+1)/2$. The following table for $n = 100$ was calculated using $C = 40$ and checked (except the last entry) using $C = 30$.

s	0	1	2	3	4	5	6	7	8	9	10
$\psi(S, 100)$....	.13731	.12943	.10841	.08066	.05328	.03124	.01624	.00748	.00305	.00110	.00035

For negative values of s the values of ψ will be the same as for positive values. The calculation is rather tedious as fifteen or twenty terms of the series have to be used. If no great accuracy is required, it is possible to use in this, and, in many other cases, certain well-known asymptotic formulas. They are taken up in the last section.

3. A Fourier's series approximating the Gaussian error curve.

The general theory of compounding frequencies leads us to suspect that a Gaussian error function would give a good approximation to the frequency in the problem of the last section and similar ones. The results of this section are developed chiefly for the purpose of taking this point up in the next section, yet they present some interest of their own. If we develop the function $a/\sqrt{\pi}\cdot e^{-a^2x^2}$ in a Fourier's series in the interval $(-C, C)$, we obtain a series of cosines alone, since the function is even. The ith coefficient is

$$a_i = \frac{2}{C}\int_0^C \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} \cos\frac{i\pi\lambda}{C}\, d\lambda.$$

This cannot be integrated in closed form. We therefore adopt an approximation by extending the upper limit to infinity, thereby making an error in each coefficient, the total effect of which is to be examined presently. Denoting by $\bar{a}_i$ this approximate value of the ith coefficient, we have*

$$\bar{a}_i = \frac{2}{C}\int_0^\infty \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} \cos\frac{i\pi\lambda}{C}\, d\lambda = \frac{1}{C} e^{-(i^2\pi^2/4a^2C^2)}.$$

Hence the approximate development of $a/\sqrt{\pi}\cdot e^{-a^2x^2}$ is

$$\frac{1}{2C} + \frac{1}{C}\sum_{i=1}^{\infty} e^{-(i^2\pi^2/4a^2C^2)} \cos\frac{i\pi x}{C}. \tag{2}$$

Now let us denote by $\epsilon(x)$ the error made by substituting this series for the Gaussian error function. Then

$$\epsilon(x) = \tfrac{1}{2}\cdot\frac{2}{C}\int_C^\infty \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} d\lambda + \sum_{i=1}^{\infty}\left[\frac{2}{C}\int_C^\infty \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2}\cos\frac{i\pi\lambda}{C}\, d\lambda\right]\cos\frac{i\pi x}{C}$$

* For this integration see B. O. Pierce, A Short Table of Integrals, p. 64, Formula 508.

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$$= \frac{a}{C\sqrt{\pi}}\left[\int_C^{2C} e^{-a^2\lambda^2}d\lambda + \int_{2C}^{3C} e^{-a^2\lambda^2}d\lambda + \cdots\right]$$

$$+ \frac{2a}{C\sqrt{\pi}}\sum_{i=1}^{\infty}\left[\int_C^{2C} e^{-a^2\lambda^2}\cos\frac{i\pi\lambda}{C}d\lambda + \int_{2C}^{3C} e^{-a^2\lambda^2}\cos\frac{i\pi\lambda}{C}d\lambda + \cdots\right]\cos\frac{i\pi x}{C}.$$

Next change the variables of integration as follows: In the first integral of each term of the sum, let $\lambda = 2C - \mu$, in the second integral of each term of the sum, let $\lambda = 2C + \mu$, in the third integral of each term of the sum, let $\lambda = 4C - \mu$, in the fourth integral of each term of the sum, let $\lambda = 4C + \mu$, and so on regularly. This is designed to reduce each set of limits to zero and C without introducing troublesome negative signs. After slight trigonometric reductions and judicious changes of limits we obtain

$$\epsilon(x) = \frac{a}{C\sqrt{\pi}}\int_0^C\left[e^{-a^2(2C-\mu)^2} + e^{-a^2(2C+\mu)^2} + \cdots\right]d\mu$$

$$+ \frac{2a}{C\sqrt{\pi}}\sum_{i=1}^{\infty}\int_0^C\left[e^{-a^2(2C-\mu)^2} + e^{-a^2(2C+\mu)^2} + \cdots\right]\cos\frac{i\pi\mu}{C}d\mu\cdot\cos\frac{i\pi x}{C}.$$

But we recognize this as the Fourier's series which represents that function of x obtained by substituting x for μ in the bracket. Hence

$$\epsilon(x) = \frac{a}{\sqrt{\pi}}\left[e^{-a^2(2C-x)^2} + e^{-a^2(2C+x)^2} + e^{-a^2(4C-x)^2} + \cdots\right].$$

Thus we have found not merely an upper limit on the error but the exact error for any value of x, given as a function of x. Since x must lie in the interval $(-C, C)$ this error is very small even for moderately large values of a and C. Its greatest value occurs when $x = C$, and its smallest value when $x = 0$. The property most valuable to us in what follows is that for a fixed value of a the error approaches zero if C is allowed to increase indefinitely.

It is rather curious that the Fourier's series (2) is equal to $a/\sqrt{\pi}\cdot e^{-a^2x^2} + \epsilon(x)$ not only in the interval $(-C, C)$ but for all values of x, since the latter function is periodic with a period of $2C$.

4. Asymptotic formulas for some frequencies.

Let us compare coefficients in series (1) and (2) for the purpose of determining a value of a which shall make the Gaussian error function approximately represent the frequency in the problem of section 2. For

this we shall use the following more or less obvious formulas:

$$e^{-(y^2/6)} = 1 - \frac{y^2}{6} + \frac{y^4}{72} - \cdots,$$

$$\frac{\sin y}{y} = 1 - \frac{y^2}{6} + \frac{y^4}{120} - \cdots,$$

$$(e^{-(y^2/6)})^n - \left(\frac{\sin y}{y}\right)^n = \left(e^{-(y^2/6)} - \frac{\sin y}{y}\right) \sum_{i=0}^{n-1} (e^{-(y^2/6)})^i \left(\frac{\sin y}{y}\right)^{n-i-1},$$

$$(3) \qquad (e^{-(y^2/6)})^n - \left(\frac{\sin y}{y}\right)^n \leqq \frac{ny^4}{72} (e^{-(y^2/6)})^{n-1} \leqq \frac{27n}{100(n-1)^2}.$$

The last inequality is obtained by taking the maximum of the preceding function of y.

Now write down the difference between the ith coefficients in series (2) and (1), changing their forms slightly to facilitate comparison with formula (3). This gives

$$\frac{1}{C} [e^{-1.6(i\pi/2C)^2}]^{6/a^2} - \frac{1}{C} \left[\frac{\sin\left(\frac{i\pi}{2C}\right)}{\left(\frac{i\pi}{2C}\right)}\right]^n.$$

If we take a to satisfy $6/a^2 = n$, or $a = \sqrt{6/n}$, then supposing n to be rather large, say twenty or more, we can simplify the maximum difference given by (3) and say that the maximum difference between any two corresponding coefficients in series (2) and (1) does not exceed $3/10\ Cn$.

Next let k denote the smallest integer greater than or equal to $2C$ and consider the remainder of series (1) after the kth term. This remainder is obviously less than the following

$$\frac{2^n C^{n-1}}{\pi^n} \sum_{i=k+1}^{\infty} \frac{1}{i^n} < \frac{2^n C^{n-1}}{\pi^n} \int_{2C}^{\infty} \frac{d\lambda}{\lambda^n} = \frac{2}{(n-1)\pi^n}.$$

The remainder of series (2) after k terms is still smaller. Now the average difference between corresponding coefficients in the first k terms in series (2) and (1) will not exceed one half the maximum, as is seen from (3), so that in the sum of these terms we cannot have an error greater than $3/10\ n$. This limit is extravagant enough to cover the error made in neglecting terms beyond the kth. Also the error discussed in section (3) can be made as small as we please by taking C large. Hence *if n is not less than twenty in the problem of section* (2), *the Gaussian error function with $a = \sqrt{6/n}$ approximates the frequency, the discrepancy being less than* $3/10\ n$. In actual computation the discrepancy seems to be much less. For the case $n = 100$ it is about $1/100\ n$.

As n grows infinite every ordinate in the graphs of both the frequency function and the Gaussian function approaches zero, so that it is evident a priori that the difference between corresponding ordinates must approach zero. This might seem to rob our result of all significance; but it will be noticed that the greatest ordinate in the Gaussian curve is $\sqrt{6}/n\pi$ so that the difference between corresponding ordinates near the origin is an infinitesimal of higher order than either of the ordinates, as n grows infinite.

Let us take up another problem, an extension of the previous. Given n variables each lying between $-\frac{1}{2}l$ and $+\frac{1}{2}l$, such that every value of each in this interval is equally probable. Find a Gaussian error function approximating the frequency of their sum. A similar course of reasoning shows that the Gaussian function with $a = \sqrt{6}/nl^2$ gives an approximation to the frequency whose greatest discrepancy is less than $3/10\ nl$. Let us vary the problem by supposing n to increase and l to diminish in such a way that nl^2 remains equal to some arbitrary constant. The approximating Gaussian function would not change, but the discrepancy would approach zero. Evidently a problem of this sort could be framed whose frequency curve closely resembled any arbitrary Gaussian error curve.

Next we shall consider the sum of n variables each having the same frequency $f(x)$, which shall be identically zero outside a certain interval $(-l, l)$, and capable of development in a Fourier's series of cosines in any larger interval $(-C, C)$. Then the frequency of the sum is approximated by the Gaussian error function in which a is determined by

$$\frac{1}{a^2} = 4n \int_0^l \lambda^2 f(\lambda) d\lambda, \tag{4}$$

and the approximation becomes better as n increases.

A sketch* of the proof is as follows: We have

$$f(x) = \frac{1}{2C} + \sum_{i=1}^{\infty} a_i \cos \frac{i\pi x}{C}.$$

The constant term is known since $f(x)$ is a frequency function and therefore its integral from $-C$ to C is unity. The frequency for the sum of the n variables is, by application of Theorem 1,

$$\frac{1}{2C} + C^{n-1} \sum_{i=1}^{\infty} a_i^n \cos \frac{i\pi x}{C}. \tag{5}$$

* Bessel has obtained this result and the following one; Astr. Nachr., vol. 15, pp. 369–405. Our only excuse for giving them here is that the use of Fourier's series in this connection seems new and interesting.

Now we know that a_i is less than A/i, where A is some constant.* Consequently the remainder beyond a certain term in series (5) will be very small. To show that the earlier terms in series (5) and (2) are nearly equal when a is chosen as above, we exhibit a_i as follows:

$$a_i = \frac{2}{C}\int_0^C f(\lambda)\cos\frac{i\pi\lambda}{C}\,d\lambda = \frac{2}{C}\int_0^l f(\lambda)\cos\frac{i\pi\lambda}{C}\,d\lambda$$

$$= \frac{2}{C}\left[\frac{1}{2} - \frac{i^2\pi^2}{2!\,C^2}\int_0^l \lambda^2 f(\lambda)d\lambda + \frac{i^4\pi^4}{4!\,C^4}\int_0^l \lambda^4 f(\lambda)d\lambda - \cdots\right].$$

The last form is obtained by developing $\cos i\pi\lambda/C$ into a power series. Now if we develop the ith coefficient in (2) into a series of powers of the exponent of e, and compare with the above, it will be seen that we have so chosen a in (4) that the coefficients of (5) and (2) are nearly equal for small values of $i\pi/C$, that is for the earlier terms of the two series. The proof can be made rigorous.

A further generalization of the problem is to consider n variables each with a different frequency function, $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$. If each of these functions is zero outside of an interval $(-l, l)$ and capable of development in a Fourier's series of cosines, then the Gaussian error function in which a is determined by

$$\frac{1}{a^2} = 4\sum_{i=1}^{n}\int_0^l \lambda^2 f_i(\lambda)d\lambda$$

will approximate the frequency in an asymptotic manner.

The question of asymptotic formulas when frequencies are compounded which are not *even* functions has been investigated by other methods. It looks as if theorem 1 might be used to establish these results also, but the process would probably be too tedious for interest.

SHEFFIELD SCIENTIFIC SCHOOL,
February 2, 1917.

* Picard, loc. cit., p. 253.

CONJUGATE SYSTEMS OF CURVES ON A SURFACE BOTH OF WHOSE LAPLACE TRANSFORMS ARE LINES OF CURVATURE.

By J. M. Stetson.

Introduction.

The first part of this paper is devoted to the study of the relations between the radial transformation and some of the other simple transformations of general conjugate systems, and later these results are applied to the surfaces which give the paper its title. Two conjugate systems are radial transforms of each other if the straight lines joining corresponding points are concurrent. It is found that if two conjugate systems are in this relation, all conjugate systems obtained by the transformations of Levy and Laplace from one conjugate system are radial transforms of the systems similarly obtained from the other. A geometric definition of the Levy transforms in terms of the radial transforms is a consequence of this fact. In the case of conjugate systems with equal invariants it is found that the Moutard transforms are associate to the radial transforms.

Among the conjugate systems whose Laplace transforms are lines of curvature, those with equal invariants are of fundamental importance, and are the only ones considered here. For convenience, they are called conjugate systems (G). It is found that they may be defined as the conjugate systems whose Laplace transforms invert by reciprocal radii into the Laplace transforms of another conjugate system. The fact that the transformation of conjugate systems (G) so suggested is a radial transformation and therefore capable of being expressed simply has hitherto escaped attention. The fact that the systems associate to conjugate systems (G) are conjugate systems (G) has not been given geometric expression before, but Calapso has given the equations of transformation that establish that fact. In view of the relations established in the first section between the radial, associate, and Moutard transforms, the Moutard transformations of conjugate systems (G), though previously unknown, are combinations of transformations whose existence has been previously shown.

In a recent paper Eisenhart* considered certain *transformations K*

* Transactions of the American Mathematical Society, vol. 15 (1914), pp. 397–430.

of conjugate systems with equal point invariants. The third part establishes the existence of transformations K which when applied to a system (G) give rise to new systems (G).

The rest of the paper is devoted to geometric properties of conjugate systems (G). Most of them are obtained by considering a congruence of spheres which is suggested by an analytic result in the second part of this paper. Guichard's original theorems about these systems were obtained from a consideration of the same congruence. One of the results is shown to be characteristic, affording a third definition of conjugate systems (G).

1. **Laplace transformations. Radial transformations.** The envelope of the tangent planes to a surface at points along a curve form a developable surface. Through each point of the curve will pass one ruling of the developable surface. The direction determined by this ruling is called *conjugate** to the direction of the curve. Two families of curves are said to form a *conjugate system* if at each point the direction of the curve of either family is conjugate to the direction of the curve of the other family which passes through that point. Every family of curves forms part of one such conjugate system, except that on every surface there exist not more than two self-conjugate systems, called *asymptotic lines*.† On every surface there is a unique orthogonal conjugate system, known as the *lines of curvature*. A necessary and sufficient condition that the parametric lines of a surface form a conjugate system is that the coördinates x, y, z, satisfy a *Laplace equation*,

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{1}$$

where a and b are functions of u and v.‡ The coördinates x, y, z, may be thought of as defining the parametric conjugate system, and will be spoken of in this paper as the coördinates of the conjugate system.

Let the tangents be drawn to the curves u = constant of a parametric conjugate system S. In the congruence thus formed one family of developable surfaces will have for edges of regression the curves u = constant of S. From the definition of conjugate directions it is obvious that the other family of developables of the congruence is given by v = constant. The edges of regression of this family form the other focal sheet of the congruence. This surface is called a *Laplace transform* of S, and the notation S_1 is used for it. The parametric curves on S_1 are conjugate. The coördinates of S_1 are given by the equations§

* Eisenhart, pp. 126, 127. A reference of this sort is to Eisenhart's Differential Geometry.

† Eisenhart, p. 128.

‡ Eisenhart, pp. 127, 154.

§ Eisenhart, pp. 403–406. The notation is different.

$$x_1 = x - \frac{1}{a}\frac{\partial x}{\partial v}, \qquad y_1 = x - \frac{1}{a}\frac{\partial y}{\partial v}, \qquad z_1 = z - \frac{1}{a}\frac{\partial z}{\partial v}. \tag{2}$$

The other focal surface of the congruence of tangents to the curves v = constant of S gives another *Laplace transform* S_{-1}, whose coördinates are given by

$$x_{-1} = x - \frac{1}{b}\frac{\partial x}{\partial u}, \qquad y_{-1} = y - \frac{1}{b}\frac{\partial y}{\partial u}, \qquad z_{-1} = z - \frac{1}{b}\frac{\partial z}{\partial u}. \tag{3}$$

The two Laplace transforms of S_1 are S and a new conjugate system S_2; those of S_{-1} are S and a new conjugate system S_{-2}. There is thus formed a *suite of Laplace*

$$\cdots S_{-m} \cdots S_{-2}, S_{-1}, S, S_1, S_2 \cdots S_n \cdots.$$

The Laplace transforms of any conjugate system S_i in the suite are the systems adjoining it, S_{i-1} and S_{i+1}. Here $S_0 = S$.

In order that the transformations

$$x' = \frac{x}{\lambda}, \qquad y' = \frac{y}{\lambda}, \qquad z' = \frac{z}{\lambda}, \tag{4}$$

where λ is a function of u and v, transform a conjugate system S whose coördinates are x, y, z, into another conjugate system S', it is necessary and sufficient that x, y, z, λ be four solutions of the same Laplace equation (1). The coördinates x', y', z' of S' will then satisfy the equation*

$$\frac{\partial^2\theta'}{\partial u \partial v} = \left(a - \frac{\partial}{\partial v}\log\lambda\right)\frac{\partial\theta'}{\partial u} + \left(b - \frac{\partial}{\partial u}\log\lambda\right)\frac{\partial\theta'}{\partial v}. \tag{5}$$

The transformation given by equations (4) under these conditions will be spoken of as a *radial transformation*, or, more explicitly, as the radial transformation determined by λ.

An equation of Laplace of the form (1) is said to have *equal invariants* if $(\partial a/\partial u) = (\partial b/\partial v)$. Equation (5) shows that all the radial transforms of S will have equal invariants when and only when S itself has equal invariants.

Let us find the Laplace transforms of S', defined by (4). The coördinates of S_1 are given by equations of the form (2). The coördinates of S_1', the Laplace transform of S', are given by equations of the form

$$x_1' = x' - \frac{1}{a - \frac{\partial}{\partial v}\log\lambda}\frac{\partial x'}{\partial v}$$

* These statements can readily be verified by substitution.

$$= \frac{x}{\lambda} - \frac{\lambda \dfrac{\partial x}{\partial v} - x \dfrac{\partial \lambda}{\partial v}}{\left(a - \dfrac{\partial}{\partial v} \log \lambda\right) \lambda^2} = \frac{x_1}{\lambda_1},$$

where

$$\lambda_1 = \lambda - \frac{1}{a} \frac{\partial \lambda}{\partial v}.$$

Observe that λ_1 is the Laplace transform of λ, as x_1 is of x. We therefore have

$$x_1' = \frac{x_1}{\lambda_1}, \qquad y_1' = \frac{y_1}{\lambda_1}, \qquad z_1' = \frac{z_1}{\lambda_1}.$$

Similarly

$$x_{-1}' = \frac{x_{-1}}{\lambda_{-1}}, \qquad y_{-1}' = \frac{y_{-1}}{\lambda_{-1}}, \qquad z_{-1}' = \frac{z_{-1}}{\lambda_{-1}}.$$

Repeated application of these formulæ gives

THEOREM I: *If S' is the radial transform of S determined by λ, then each member of the suite of Laplace of S' is obtained from the corresponding member of the suite of Laplace of S by the radial transformation determined by the corresponding member of the suite of Laplace of λ.*

2. **Transformations of Levy.** Levy* has shown that every surface $\overline{S}$ which is cut in a conjugate system by the developables of the congruence of tangents to the curves u = constant of a conjugate system S has the coördinates

$$\bar{x} = x - \frac{\mu}{\dfrac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v}, \qquad \bar{y} = y - \frac{\mu}{\dfrac{\partial \mu}{\partial v}} \frac{\partial y}{\partial v}, \qquad \bar{z} = z - \frac{\mu}{\dfrac{\partial \mu}{\partial v}} \frac{\partial z}{\partial v}, \tag{6}$$

where μ is a solution of the Laplace equation (1) satisfied by x, y, z. In the same way, surfaces $\overline{S}_{-1}$ which are cut in a conjugate system by the developables of the congruence of tangents to the curves v = constant of S have coördinates

$$\bar{x}_{-1} = x - \frac{\mu}{\dfrac{\partial \mu}{\partial u}} \frac{\partial x}{\partial u}, \qquad \bar{y}_{-1} = y - \frac{\mu}{\dfrac{\partial \mu}{\partial u}} \frac{\partial y}{\partial u}, \qquad \bar{z}_{-1} = z - \frac{\mu}{\dfrac{\partial \mu}{\partial u}} \frac{\partial z}{\partial u}. \tag{7}$$

If the same solution μ of (1) be used in (6) and (7), the lines joining corresponding points of $\overline{S}$ and $\overline{S}_{-1}$ are tangent to the curves v = constant of $\overline{S}$ and to the curves u = constant of $\overline{S}_{-1}$. The surfaces $\overline{S}$ and $\overline{S}_{-1}$ are therefore Laplace transforms of each other, and the notation used is justified.

* "Sur quelques équations linéaires aux derivées partielles," Journal de l'École Polytechnique, cahier 56 (1886), p. 77. Cf. also Darboux, *Leçons*, vol. 2, chap. x.

The congruence of tangents to the curves u = constant of S is identical with the congruence of tangents to the curves v = constant of S_1. The coördinates of $\bar{S}$ must therefore be expressible in the form

$$\bar{x} = x_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial x_1}{\partial u}, \qquad \bar{y} = y_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial y_1}{\partial u}, \qquad \bar{z} = z_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial z_1}{\partial u},$$

where x_1, y_1, z_1, are the coördinates of S_1, and μ_1 satisfies the Laplace equation of S_1. In fact, we have

$$\bar{x} = x - \frac{\mu}{\frac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v} = x - \frac{1}{a} \frac{\partial y}{\partial v} - \frac{\mu - \frac{1}{a} \frac{\partial \mu}{\partial v}}{\frac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v}.$$

But

$$\frac{\partial x_1}{\partial u} = \frac{1}{a} \left(\frac{\partial}{\partial u} \log a - b \right) \frac{\partial x}{\partial v}$$

and we take μ_1 defined by

$$\mu_1 = \mu - \frac{1}{a} \frac{\partial \mu}{\partial v},$$

$$\frac{\partial \mu_1}{\partial u} = \frac{1}{a} \left(\frac{\partial}{\partial u} \log a - b \right) \frac{\partial \mu}{\partial v},$$

so that μ_1 is the Laplace transform of μ. Then

$$\bar{x} = x - \frac{\mu}{\frac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v} = x_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial u}} \frac{\partial x_1}{\partial u}, \qquad \text{etc.},$$

and by the theorem of Levy the coördinates of $\bar{S}_1$, the Laplace transform of $\bar{S}$, are given by

$$\bar{x}_1 = x_1 - \frac{\mu_1}{\frac{\partial \mu_1}{\partial v}} \frac{\partial x_1}{\partial v}, \qquad \text{etc.} \tag{8}$$

In like manner

$$\bar{x}_{-1} = x - \frac{\mu}{\frac{\partial \mu}{\partial u}} \frac{\partial x}{\partial u} = x_{-1} - \frac{\mu_{-1}}{\frac{\partial \mu_{-1}}{\partial v}} \frac{\partial x_{-1}}{\partial v}, \qquad \text{etc.}, \tag{9}$$

and

$$\bar{x}_{-2} = x_{-1} - \frac{\mu_{-1}}{\frac{\partial \mu_{-1}}{\partial u}} \frac{\partial x_{-1}}{\partial u}.$$

If we call $\bar{S}$, $\bar{S}_{-1}$ the *Levy transforms of S determined by μ*, repeated application of (6), (7), (8), (9) gives

THEOREM II: *If $\bar{S}$, $\bar{S}_{-1}$ are the Levy transforms of S determined by μ, then their nth Laplace transforms $\bar{S}_n$, $\bar{S}_{n-1}$ are the Levy transforms of S_n, the nth Laplace transform of S, determined by μ_n, the nth Laplace transform of μ.*

Consider now the conjugate systems $\bar{S}'$, $\bar{S}_{-1}'$, the Levy transforms of S' determined by $\mu' = \mu/\lambda$, which evidently is a solution of the Laplace equation (5) satisfied by the coördinates of S'. We have

$$\bar{x}' = x' - \frac{\mu'}{\dfrac{\partial \mu'}{\partial v}} \frac{\partial x'}{\partial v} = \frac{x}{\lambda} - \frac{\dfrac{\mu}{\lambda}}{\dfrac{\partial}{\partial v}\left(\dfrac{\mu}{\lambda}\right)} \frac{\partial}{\partial v}\left(\frac{x}{\lambda}\right)$$

$$= \frac{x}{\bar{\lambda}} - \frac{\mu\lambda\left(\lambda \dfrac{\partial x}{\partial v} - x\dfrac{\partial \lambda}{\partial v}\right)}{\left(\lambda \dfrac{\partial \mu}{\partial v} - \mu \dfrac{\partial \lambda}{\partial v}\right)\lambda^2} = \frac{1}{\lambda - \dfrac{\mu}{\dfrac{\partial \mu}{\partial v}} \dfrac{\partial \lambda}{\partial v}}\left(x - \frac{\mu}{\dfrac{\partial \mu}{\partial v}} \frac{\partial x}{\partial v}\right)$$

$$= \frac{\bar{x}}{\bar{\lambda}},$$

where $\bar{\lambda}$ is the Levy transform of λ determined by μ. Similarly we have

$$\bar{x}_{-1}' = \frac{\bar{x}_{-1}}{\bar{\lambda}_{-1}}.$$

Repetition of these results gives

THEOREM III: *If S' is the radial transform of S determined by λ, the Levy transforms of S' determined by μ/λ are radial transforms of the Levy transforms of S determined by μ; and the solutions which determine the radial transformations are the corresponding Levy transforms of λ.*

If μ is a solution of (1), so is $1 - \mu$. The Levy transforms of $1 - \mu$ determined by μ are

$$1 - \mu - \frac{\mu}{\dfrac{\partial \mu}{\partial v}} \frac{\partial}{\partial v}(1 - \mu) = 1$$

and

$$1 - \mu - \frac{\mu}{\dfrac{\partial \mu}{\partial u}} \frac{\partial}{\partial u}(1 - \mu) = 1.$$

We have therefore as a consequence of the preceding theorem,

THEOREM IV: *Tangents at corresponding points to the curves $u =$ constant of a surface and its radial transform determined by $1 - \mu$ intersect at the corresponding point of the Levy transform determined by μ.*

The corresponding statement with regard to the curves v = constant is true.

3. Associate surfaces. Transformations of Moutard. Two surfaces, S and S_0, are said to be *associate* if the planes at corresponding points are parallel, and to the asymptotic lines of either surface there corresponds a conjugate system on the other. If any correspondence is established between two surfaces, there will be in general one and only one system of corresponding curves which is conjugate on both surfaces.* This system is called the corresponding conjugate system. In the case of associate surfaces, it can be shown† that if the corresponding conjugate system is parametric, it has equal invariants and the coördinates of S_0 are given by the quadratures of the form

$$\frac{\partial x_0}{\partial u} = \lambda \frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = -\lambda \frac{\partial x}{\partial v}, \tag{10}$$

where λ is given by the equations

$$\frac{\partial \log \lambda}{\partial u} = -2b, \qquad \frac{\partial \log \lambda}{\partial v} = -2a. \tag{11}$$

It will be convenient to speak of the corresponding conjugate systems of S and S_0 as associate conjugate systems.

Since the Laplace equation must have equal invariants, we may write

$$a = \frac{\partial \varphi}{\partial v}, \qquad b = \frac{\partial \varphi}{\partial u}, \qquad \lambda = e^{-2\phi}. \tag{12}$$

Hence

$$\frac{\partial x_0}{\partial u} = e^{-2\phi} \frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = -e^{-2\phi} \frac{\partial x}{\partial v}, \tag{13}$$

and similar equations in y_0 and z_0.

If x and θ are two solutions of the equation

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \varphi}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \varphi}{\partial u}\frac{\partial \theta}{\partial v},$$

the conditions of integrability of the following equations are satisfied:

$$\frac{\partial \xi}{\partial u} = e^{-2\phi}\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = -e^{-2\phi}\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right). \tag{14}$$

It can readily be shown that

$$\frac{\partial^2 \xi}{\partial u \partial v} = \frac{\partial}{\partial v}(\log \theta - \varphi)\frac{\partial \xi}{\partial u} + \frac{\partial}{\partial u}(\log \theta - \varphi)\frac{\partial \xi}{\partial v}. \tag{15}$$

* Eisenhart, p. 130.

† Eisenhart, pp. 378–380.

Hence the function ξ is a solution of a Laplace equation with equal invariants. Equations (14) define the *transformation of Moutard.* The conjugate system whose coördinates are given by ξ, η, ζ, where η and ζ are defined by analogous equations in y and z, is called the *Moutard transform* determined by the solution θ.

Equations (14) can be written

$$\frac{\partial \xi}{\partial u} = e^{-2\phi}\theta^2 \frac{\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right)}{\theta^2},$$

$$\frac{\partial \xi}{\partial u} = -\, e^{-2\phi}\theta^2 \frac{\partial x'}{\partial u}, \tag{16}$$

where

$$x' = \frac{x}{\theta}.$$

Similarly

$$\frac{\partial \xi}{\partial v} = e^{-2\phi}\theta^2 \frac{\partial x'}{\partial v}, \tag{17}$$

and similar equations connect η and y', ζ and z'.

Since the Laplace equation, satisfied by x', y', z', is

$$\frac{\partial^2 \theta'}{\partial u \partial v} = \frac{\partial}{\partial v}(\varphi - \log\theta)\frac{\partial \theta'}{\partial u} + \frac{\partial}{\partial u}(\varphi - \log\theta)\frac{\partial \theta'}{\partial v},$$

equations (16) and (17) show that the Moutard transform is a conjugate system associate to a radial transform, and we have

THEOREM V: *The Moutard transforms of a conjugate system are conjugate systems associate to its radial transforms; and the solution θ which determines the Moutard transform determines also the radial transform to which it is associate.*

Similarly, the radial transformation determined by θ followed by a Moutard transformation determined by $1/\theta$ gives a conjugate system associate to the original conjugate system.

4. **Conjugate systems** (G). Conjugate systems whose Laplace transforms in both senses are lines of curvature have been studied by Guichard* and Calapso.† It is found that if (1) is the Laplace equation of the original conjugate system S, the fundamental quantities of the first order,‡

* "Sur les réseaux qui, par la méthode de Laplace, se transforment des deux côtés en réseaux orthogonaux," Comptes Rendus, vol. 132, 1901, p. 249.

† "Un problema sui sistemi di linee fra loro coniugate e sulle relative trasformazioni di Laplace," Annali di Matematica, serie III, tomo XIII (1907), p. 203.

‡ Eisenhart, p. 71.

E and G, of the surface S, are given by

$$E = e^{2\psi}\left(\frac{\partial \psi}{\partial u}\right)^2, \qquad G = e^{2\phi}\left(\frac{\partial \varphi}{\partial v}\right)^2, \tag{18}$$

where φ and ψ satisfy the equations

$$\frac{\partial \varphi}{\partial v} = a, \qquad \frac{\partial \psi}{\partial u} = b. \tag{19}$$

Conversely, any conjugate system for which E and G are so related to the coefficients of the Laplace equation has lines of curvature for its Laplace transforms.

The conjugate systems with equal invariants, for which $\varphi = \psi$, are found to be of fundamental importance. They will be called in this paper *conjugate systems* (G).* For a conjugate system (G) the fundamental quantities of the first order are

$$E = e^{2\phi}\left(\frac{\partial \varphi}{\partial u}\right)^2, \qquad F = e^{2\phi}\frac{\partial^2 \varphi}{\partial u \partial v}, \qquad G = e^{2\phi}\left(\frac{\partial \varphi}{\partial v}\right)^2, \tag{20}$$

and the Laplace equation is

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \varphi}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \varphi}{\partial u}\frac{\partial \theta}{\partial v}.\dagger \tag{21}$$

If the Laplace transforms, S_1 and S_{-1}, of any conjugate system S which satisfies the Laplace equation (1) are lines of curvature, $x_1^2+y_1^2+z_1^2$ is a solution of the Laplace equation of S_1, and $x_{-1}^2 + y_{-1}^2 + z_{-1}^2$ is a solution of the Laplace equation of S_{-1}. There must then be a solution λ of (1) whose Laplace transform λ_1 is $x_1^2 + y_1^2 + z_1^2$, and a solution μ whose Laplace transform μ_{-1} is $x_{-1}^2 + y_{-1}^2 + z_{-1}^2$.

We have

$$\lambda_1 = \lambda - \frac{1}{a}\frac{\partial \lambda}{\partial v}. \tag{22}$$

Differentiating and remembering that λ is a solution of (1), we have

$$\frac{\partial \lambda_1}{\partial u} = \frac{\frac{\partial a}{\partial u} - ab}{a^2}\frac{\partial \lambda}{\partial v}. \tag{23}$$

If we give λ_1, x_1, y_1, z_1 their values in terms of λ, x, y, z, the equations (22)

* Calapso uses the same notation in a less restricted sense.

† The function φ is not arbitrary, but must satisfy a differential equation of the fourth order as Calapso has shown.

and (23) give

$$\lambda = x^2 + y^2 + z^2 - \frac{1}{a^2}\left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Remembering that the numerator of the fraction is $G = e^{2\phi}(\partial\varphi/\partial v)^2$ and $a = \partial\varphi/\partial v$, we obtain

$$\lambda = x^2 + y^2 + z^2 - e^{2\phi}.$$

In like manner we obtain

$$\mu = x^2 + y^2 + z^2 - e^{2\psi}.$$

Evidently $\mu = \lambda$ only when $\varphi = \psi$; that is, when S is a conjugate system (G).

This result can be interpreted geometrically. The radial transformation,

$$x_1' = \frac{x_1}{x_1^2 + y_1^2 + z_1^2}, \quad \text{etc.},$$

is an inversion by reciprocal radii; the transform S_1' it gives is the Laplace transform of the radial transform of S, determined by λ. In the same way the transform S_{-1}' of S_{-1} by reciprocal radii is a Laplace transform of the radial transform of S, determined by μ. It is apparent that for S_1', S_{-1}' to be Laplace transforms of the same conjugate system S', λ and μ must be equal. Since the transform by reciprocal radii of a conjugate system Σ is not a conjugate system unless Σ is a system of lines of curvature, we have

THEOREM VI: *The conjugate systems (G) are the conjugate systems whose Laplace transforms invert by reciprocal radii into the Laplace transforms of another conjugate system.*

It is to be noticed that in an inversion by reciprocal radii, not only are S_1' and S_{-1}' the transforms of S_1 and S_{-1}, but S_1 and S_{-1} are the transforms of S_1' and S_{-1}'. In the case of conjugate systems (G), then, S' is a conjugate system whose Laplace transforms invert into the Laplace transforms of another conjugate system. We have proved

THEOREM VII: *If S is a conjugate system (G), S', its radial transform determined by the solution $x^2 + y^2 + z^2 - e^{2\phi}$, is also a conjugate system (G); and the Laplace transforms of either conjugate system transform by an inversion by reciprocal radii into the Laplace transforms of the other.*

The conjugate systems associate to a conjugate system (G) are defined by equations (10). The fundamental quantities are therefore

$$E_0 = e^{-4\phi}E = e^{-2\phi}\left(\frac{\partial\varphi}{\partial u}\right)^2, \qquad G_0 = e^{-2\phi}\left(\frac{\partial\varphi}{\partial v}\right)^2.$$

Since the Laplace equation of the associate conjugate systems is

$$\frac{\partial^2\theta}{\partial u\partial v} = -\frac{\partial\varphi}{\partial v}\frac{\partial\theta}{\partial u} - \frac{\partial\varphi}{\partial u}\frac{\partial\theta}{\partial v},$$

they will be conjugate systems (G) if

$$E_0 = e^{2\omega}\left(\frac{\partial\omega}{\partial u}\right)^2, \qquad G_0 = e^{2\omega}\left(\frac{\partial\omega}{\partial v}\right)^2,$$

where

$$\frac{\partial\omega}{\partial u} = -\frac{\partial\varphi}{\partial u}, \qquad \frac{\partial\omega}{\partial v} = -\frac{\partial\varphi}{\partial v}.$$

These conditions are satisfied by $\omega = (\log c - \varphi)$. We have proved

THEOREM VIII: *The conjugate systems associate to a conjugate system (G) are conjugate systems (G).*

On account of Theorem V we have

THEOREM IX: *The Moutard transforms of a conjugate system (G) determined by $x^2 + y^2 + z^2 - e^{2\phi}$ are themselves conjugate systems (G).*

5. **Transformations K.** In a recent paper, Eisenhart* has considered transformations of conjugate systems with equal invariants. These he calls *transformations K*. The K-transform of a conjugate system S is the radial transform determined by λ of the Moutard transform determined by θ, where λ satisfies the equations

$$\frac{\partial\lambda}{\partial u} = -e^{-2\phi}\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\lambda}{\partial v} = e^{-2\phi}\frac{\partial\theta}{\partial v}. \tag{24}$$

We recall some of Eisenhart's results. If we write

$$\rho = e^{-2\phi}, \tag{25}$$

the Laplace equation of S becomes

$$\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial}{\partial v}\log\sqrt{\rho}\frac{\partial\theta}{\partial u} + \frac{\partial}{\partial u}\log\sqrt{\rho}\frac{\partial\theta}{\partial v} = 0, \tag{26}$$

and equations (24) are

$$\frac{\partial\lambda}{\partial u} = -\rho\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\lambda}{\partial v} = \rho\frac{\partial\theta}{\partial v}. \tag{27}$$

If X, Y, Z, denote the direction-cosines of the normals to the surface S, and X_1, Y_1, Z_1; X_2, Y_2, Z_2, the direction-cosines of the bisectors of the angles between the curves of the conjugate system S, the transformation

* "Transformations of conjugate systems with equal point invariants," Transactions of the American Mathematical Society, vol. XV, no. 4, pp. 397–430.

K may be defined by equations of the form

$$x_1 - x = \frac{1}{\lambda}(aX_1 + bX_2 + wX). \tag{28}$$

Writing 2ω for the angle between the curves of the conjugate system S, the quantities a, b, w are solutions of the completely integrable set of equations

$$\begin{aligned}
\frac{\partial a}{\partial u} &= -\lambda\sqrt{E}\cos\omega + b\left(\sqrt{\frac{E}{G}}\frac{\partial \log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right) + \frac{wD}{2\sqrt{E}\cos\omega} \\
&\qquad + \rho\theta\cos\omega\sqrt{E}, \\
\frac{\partial a}{\partial v} &= -\lambda\sqrt{G}\cos\omega - b\left(\sqrt{\frac{G}{E}}\frac{\partial \log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right) + \frac{wD''}{2\sqrt{E}\cos\omega} \\
&\qquad - \rho\theta\cos\omega\sqrt{G}, \\
\frac{\partial b}{\partial u} &= \lambda\sqrt{E}\sin\omega - a\left(\sqrt{\frac{E}{G}}\frac{\partial \log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right) - \frac{wD}{2\sqrt{E}\sin\omega} \\
&\qquad - \rho\theta\sin\omega\sqrt{E}, \qquad (29) \\
\frac{\partial b}{\partial v} &= -\lambda\sqrt{G}\sin\omega + a\left(\sqrt{\frac{G}{E}}\frac{\partial \log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right) + \frac{wD''}{2\sqrt{G}\sin\omega} \\
&\qquad - \rho\theta\sin\omega\sqrt{G}, \\
\frac{\partial w}{\partial u} &= -\frac{D}{2\sqrt{E}}\left(\frac{a}{\cos\omega} - \frac{b}{\sin\omega}\right), \\
\frac{\partial w}{\partial v} &= -\frac{D''}{2\sqrt{G}}\left(\frac{a}{\cos\omega} + \frac{b}{\sin\omega}\right).
\end{aligned}$$

If we write $T^2 = a^2 + b^2 + w^2$, T is a solution of the equations

$$\begin{aligned}
T\frac{\partial T}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}(a\cos\omega - b\sin\omega), \\
T\frac{\partial T}{\partial v} &= -(\rho\theta - \lambda)\sqrt{G}(a\cos\omega + b\sin\omega).
\end{aligned} \tag{30}$$

The coördinates x_1, y_1, z_1, of the K-transform satisfy equations

$$\frac{\partial x_1}{\partial u} = \frac{\rho}{\lambda^2}\left[\left(a\frac{\partial\theta}{\partial u} + \theta\lambda\cos\omega\sqrt{E}\right)X_1\right.$$

$$+\left(b\frac{\partial\theta}{\partial u}-\theta\lambda\sin\omega\sqrt{E}\right)X_2+w\frac{\partial\theta}{\partial u}X\Big],$$

$$\frac{\partial x_1}{\partial v}=-\frac{\rho}{\lambda^2}\Big[\left(a\frac{\partial\theta}{\partial v}+\theta\lambda\cos\omega\sqrt{G}\right)X_1 \tag{31}$$

$$+\left(b\frac{\partial\theta}{\partial v}+\theta\lambda\sin\omega\sqrt{G}\right)X_2+w\frac{\partial\theta}{\partial v}X\Big].$$

The Laplace equation satisfied by x_1, y_1, z_1, is

$$\frac{\partial^2\theta'}{\partial u\partial v}=\frac{\partial}{\partial v}\log\frac{\lambda}{\sqrt{\rho}\theta}\frac{\partial\theta'}{\partial u}+\frac{\partial}{\partial u}\log\frac{\lambda}{\sqrt{\rho}\theta}\frac{\partial\theta'}{\partial v}. \tag{32}$$

We wish to find transformations K which will transform a conjugate system (G) into another conjugate system (G). This will be done if the values of E_1 and G_1 obtained by squaring and adding equations like (31) are the same as those obtained from equation (32) in the same way as (20) are obtained from (21).

The value of E_1 from (31) is

$$E_1=\frac{\rho^2}{\lambda^4}\Big[\left(a\frac{\partial\theta}{\partial v}+\theta\lambda\cos\omega\sqrt{E}\right)^2+\left(b\frac{\partial\theta}{\partial u}-\theta\lambda\sin\omega\sqrt{E}\right)^2+w^2\left(\frac{\partial\theta}{\partial u}\right)^2\Big]$$

$$=\frac{\rho^2}{\lambda^4}\Big[(a^2+b^2+w^2)\left(\frac{\partial\theta}{\partial u}\right)^2+2\theta\lambda\frac{\partial\theta}{\partial u}\sqrt{E}(a\cos\omega-b\sin\omega)+\theta^2\lambda^2E\Big]. \tag{33}$$

The value of E_1 deduced from (32) is

$$E_1=\frac{\rho\theta^2}{\lambda^2}\left(\frac{\partial}{\partial u}\log\frac{\sqrt{\rho}\theta}{\lambda}\right)^2. \tag{34}$$

Since $e^{-2\phi}=\rho$ and $(\partial\lambda/\partial u)=-\rho(\partial\theta/\partial u)$, this reduces to

$$E_1=\frac{\rho\theta^2}{\lambda^2}\left[\frac{\partial\theta}{\partial u}\left(\frac{1}{\theta}+\frac{\rho}{\lambda}\right)-\frac{\partial\varphi}{\partial u}\right]^2. \tag{35}$$

When we equate the two values for E_1, we obtain, after simplification, the equation

$$\frac{\partial\theta}{\partial u}[(\lambda+\rho\theta)^2-\rho T^2]-2\frac{\partial\varphi}{\partial u}\theta\lambda[(\lambda+\rho\theta)+\sqrt{\rho}(a\cos\omega-b\sin\omega)]=0. \tag{36}$$

Similarly, (31) gives for G_1 the value

$$G_1=\frac{\rho^2}{\lambda^4}\Big[T^2\left(\frac{\partial\theta}{\partial v}\right)^2+2\theta\lambda\frac{\partial\theta}{\partial v}\sqrt{G}(a\cos\omega+b\sin\omega)+\theta^2\lambda^2G\Big], \tag{37}$$

and from (32) we deduce, using (27),

$$G_1 = \frac{\rho\theta^2}{\lambda^2}\left[\frac{\partial\theta}{\partial v}\left(\frac{1}{\theta} - \frac{\rho}{\lambda}\right) - \frac{\partial\varphi}{\partial v}\right]^2. \tag{38}$$

Equating these two values, we obtain the equation

$$\frac{\partial\theta}{\partial v}[(\lambda - \rho\theta)^2 - \rho T^2] - 2\frac{\partial\varphi}{\partial v}\theta\lambda[(\lambda - \rho\theta) + \sqrt{\rho}(a\cos\omega + b\sin\omega)] = 0. \tag{39}$$

Since $\cos 2\omega = F/\sqrt{EG}$, we have the relation

$$\frac{\partial^2\varphi}{\partial u\partial v} = \frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\cos 2\omega. \tag{40}$$

If we differentiate equation (36) with respect to v, and simplify the results by the use of equations (25), (26), (27), (30), and (40), we obtain the equation

$$\begin{aligned}&\frac{\partial\theta}{\partial u}\frac{\partial\varphi}{\partial v}[(\lambda + \rho\theta)^2 + \rho T^2 + 2\sqrt{\rho}(\lambda + \rho\theta)(a\cos\omega + b\sin\omega - 2\sqrt{\rho}\theta)]\\ &- \frac{\partial\theta}{\partial v}\frac{\partial\varphi}{\partial u}[(\lambda + \rho\theta) + \rho T^2 + 2\sqrt{\rho}(\lambda + \rho\theta)(a\cos\omega - b\sin\omega + 4\rho\theta\lambda)] \qquad (41)\\ &+ 4\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\sqrt{\rho}\theta\lambda(\sqrt{\rho}\theta - b\sin\omega) + 4\rho(\lambda + \rho\theta)\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} = 0.\end{aligned}$$

A similar process applied to (39) gives the equation

$$\begin{aligned}&\frac{\partial\theta}{\partial u}\frac{\partial\varphi}{\partial v}[(\lambda - \rho\theta)^2 + \rho T^2 + 2\sqrt{\rho}(\lambda - \rho\theta)(a\cos\omega + b\sin\omega) - 4\rho\theta\lambda]\\ &- \frac{\partial\theta}{\partial v}\frac{\partial\varphi}{\partial u}[(\lambda - \rho\theta)^2 + \rho T^2 + 2\sqrt{\rho}(\lambda - \rho\theta)(a\cos\omega - b\sin\omega + 2\sqrt{\rho}\theta)] \qquad (42)\\ &+ 4\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\sqrt{\rho}\theta\lambda(\sqrt{\rho}\theta - b\sin\omega) + 4\rho(\lambda - \rho\theta)\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} = 0.\end{aligned}$$

If the values of $\partial\theta/\partial u$, $\partial\theta/\partial v$, given by (36) and (39) be substituted in (41) and (42), these equations are identically satisfied. Hence (29), (36) and (39) form a completely integrable system. We have proved

THEOREM X: *If a, b, w, θ, satisfy the completely integrable system* (29), (36), (39), *and if λ be determined by the quadratures* (27), *the transformations K determined by θ and λ will transform the given conjugate system (G) into another conjugate system (G).*

6. Geometric properties of systems (G). Let the point M, whose coördinates are x, y, z, lie on S, a conjugate system (G), and let M_1 and M_{-1}

be the corresponding points on S_1 and S_{-1}, the Laplace transforms of S. Then the length of MM_1 is given by

$$\overline{MM}_1^2 = \Sigma(x - x_1)^2$$

$$= \sum \left[x - \left[x - \frac{\frac{\partial x}{\partial v}}{\frac{\partial \varphi}{\partial v}} \right] \right]^2 = \sum \left[\frac{\frac{\partial x}{\partial v}}{\frac{\partial \varphi}{\partial v}} \right]^2 = \frac{G}{\left(\frac{\partial \varphi}{\partial v} \right)^2} = e^{2\phi}.$$

Hence $\overline{MM}_1 = e^{\phi}$. Similarly $MM_{-1} = e^{\phi}$, and we have proved

THEOREM XI: *The lines joining a point on a conjugate system (G) with the corresponding points of its Laplace transforms are equal and have the length* e^{ϕ}.

We have seen that $x^2 + y^2 + z^2 - e^{2\phi}$ is a solution of the Laplace equation of the conjugate system (G). It is well known that when the Laplace equation of a conjugate system has a solution of the form $x^2 + y^2 + z^2 - R^2$, the line joining corresponding points on the two sheets of the envelope of the sphere whose center is at the point (x, y, z) of S and whose radius is R describes a congruence whose developables correspond to the conjugate system S.* In the case under consideration, $R = e^{\phi}$. We have just seen that the points M_1, M_{-1} lie on the sphere. The equation of the sphere is

$$(X - x)^2 + (Y - y)^2 + (Z - z)^2 - e^{2\phi} = 0. \tag{43}$$

The line joining corresponding points on the two sheets of the envelope of this sphere is given by the equations

$$(X - x)\frac{\partial x}{\partial u} + (Y - y)\frac{\partial y}{\partial u} + (Z - z)\frac{\partial z}{\partial u} + \frac{\partial \varphi}{\partial u} e^{2\phi} = 0, \tag{44}$$

$$(X - x)\frac{\partial x}{\partial v} + (Y - y)\frac{\partial y}{\partial v} + (Z - z)\frac{\partial z}{\partial v} + \frac{\partial \varphi}{\partial v} e^{2\phi} = 0. \tag{45}$$

For convenience we indicate equation (44), by $A = 0$, and (45) by $B = 0$. Equations (44) and (45), on account of (1), are the equations of the focal planes of the congruence of their intersection, which we will call the congruence g.

The point M_{-1} satisfies the equation $A = 0$; and the line MM_{-1} has direction-cosines proportional to $\partial x/\partial u$, $\partial y/\partial u$, $\partial z/\partial u$. Since MM_{-1} is a radius of the sphere (43), it follows that $A = 0$ is the equation of the tangent plane to this sphere at the point M_{-1}. Similarly, the plane $B = 0$ is tangent to the sphere at the point M_1.

* Darboux, vol. ii, p. 324.

Since $A = 0$, $B = 0$, are the focal planes of the congruence g, the focal points F_A and F_B of the congruence are given* by the two systems of equations

$$A = 0, \quad B = 0, \quad A_{vv} = 0,$$

$$A = 0, \quad B = 0, \quad B_{uu} = 0,$$

where A_{vv} indicates the second partial derivative of A with respect to v, and B_{uu} the second partial derivative of B with respect to u.

It is found that

$$A_{vv} = \Sigma\left[\left(\frac{\partial\varphi}{\partial u}\frac{\partial^2 x}{\partial v^2} + \frac{\partial^2\varphi}{\partial u\partial v}\frac{\partial x}{\partial v} + \frac{\partial^2\varphi}{\partial v^2}\frac{\partial x}{\partial u} + \frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v}\frac{\partial x}{\partial v} + \left(\frac{\partial\varphi}{\partial v}\right)^2\frac{\partial x}{\partial u}\right)(X-x)\right]$$

$$+e^{2\phi}\left[2\frac{\partial^2\varphi}{\partial v^2}\frac{\partial\varphi}{\partial u} + 2\frac{\partial\varphi}{\partial v}\frac{\partial^2\varphi}{\partial u\partial v} + 4\left(\frac{\partial\varphi}{\partial v}\right)^2\frac{\partial\varphi}{\partial u} - \frac{\partial\varphi}{\partial u}\left(\frac{\partial\varphi}{\partial v}\right)^2 - \frac{\partial\varphi}{\partial v}\frac{\partial^2\varphi}{\partial u\partial v}\right] = 0.$$

The use of $A = 0$ and $B = 0$ reduces this to the form

$$C = \Sigma\frac{\partial^2 x}{\partial v^2}(X - x) + e^{2\phi}\left[\left(\frac{\partial\varphi}{\partial v}\right)^2 + \frac{\partial^2\varphi}{\partial v^2}\right] = 0.$$

The equation $B_{uu} = 0$ in like manner may be replaced by the equation

$$D = \Sigma\frac{\partial^2 x}{\partial u^2}(X - x) + e^{2\phi}\left[\left(\frac{\partial\varphi}{\partial u}\right)^2 + \frac{\partial^2\varphi}{\partial u^2}\right] = 0.$$

We have the relation

$$\Sigma\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial v^2} = \frac{1}{2}\frac{\partial}{\partial v}\Sigma\left(\frac{\partial x}{\partial v}\right)^2 = \frac{1}{2}\frac{\partial}{\partial v}G = e^{2\phi}\left[\frac{\partial^2\varphi}{\partial v^2} + \left(\frac{\partial\varphi}{\partial v}\right)^2\right]\frac{\partial\varphi}{\partial v}.$$

Therefore the point M_1 lies on the plane $C = 0$; it also lies on the plane $B = 0$. The focal point F_A also lies on both these planes. Therefore the line M_1F_A is given by the equations

$$(X - x)\frac{\partial x}{\partial v} + (Y - y)\frac{\partial y}{\partial v} + (Z - z)\frac{\partial z}{\partial v} + e^{2\phi}\frac{\partial\varphi}{\partial v} = 0,$$

$$(X - x)\frac{\partial^2 x}{\partial v^2} + (Y - y)\frac{\partial^2 y}{\partial v^2} + (Z - z)\frac{\partial^2 z}{\partial v^2} + e^{2\phi}\left[\frac{\partial^2\varphi}{\partial v^2} + \left(\frac{\partial\varphi}{\partial v}\right)^2\right] = 0.$$

The direction cosines of this line are proportional to the determinants of the matrix

$$\begin{Vmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \end{Vmatrix}.$$

* Since the focal point is the point where the focal plane osculates the edge of regression of the developable surface it generates. See Eisenhart, p. 60.

But these are the direction-cosines of the osculating plane of the curve u = constant of S, and therefore the direction cosines of the normal to the surface S_1. Similar reasoning shows that the other focal point F_B lies on the normal to the surface S_{-1}.

The planes normal to a curve have as their envelope a developable surface whose rulings are called the *polar lines* of the curve. The polar line for a given point of a curve meets the osculating plane of the curve at that point at the center of the *osculating circle* of the curve; it meets the edge of regression of the developable at the center of the *osculating sphere* of the curve.*

Consider again the line defined by the system of equations $A = 0$, $B = 0$. On account of equation (1), this system is equivalent to the system $A = 0$, $A_v = 0$, where A_v is the partial derivative of A with respect to v. Or it is equivalent to $B = 0$, $B_u = 0$; where B_u is the partial derivative of B with respect to u. But $A = 0$ is the equation of the planes normal to the curves u = constant of S_{-1}; the equations $A = 0$, $A_v = 0$, give the polar lines of these curves. Similarly, $B = 0$, $B_u = 0$, give the polar lines of the curves v = constant of S_1. Again, the focal point F_A was defined by the system of equations

$$A = 0, \qquad B = 0, \qquad A_{vv} = 0,$$

which is equivalent to the system

$$A = 0, \qquad A_v = 0, \qquad A_{vv} = 0.$$

These equations define the centers of the osculating sphere of the curves u = constant of S_{-1}. In the same way, F_B is the center of the osculating sphere of the curve v = constant of S_1.

The tangent plane to S is the osculating plane of the curves v = constant of S_1 and of the curves u = constant of S_{-1}. The lines of the congruence g are the polar lines of these curves; their intersection with the osculating plane is the center of the osculating circle. Since the point of intersection is clearly equidistant from M_1 and M_{-1}, the osculating circles for the two systems of curves coincide.

The intersection of the normal to a surface with the polar line of a curve on a surface is the *center of normal curvature*† of the curve.

We collect the preceding results into

THEOREM XII: *If at each point of a conjugate system (G) a sphere of radius e^{ϕ} be drawn,*

1°. *The corresponding points of the Laplace transforms lie on the sphere.*

* Eisenhart, pp. 15, 38, 65.

† Eisenhart, pp. 117, 118.

2°. *The lines joining corresponding points of the two sheets of the envelope of the sphere form a congruence g, whose developables correspond to the conjugate system* (G).

3°. *The focal planes of the congruence are the tangent planes to the sphere at the points where the tangent lines of the conjugate system* (G) *cut the sphere.*

4°. *The lines of the congruence may also be defined as the common polar lines of the curves* $v = \text{constant}$ *of* S_1 *and the curves* $u = \text{constant}$ *of* S_{-1}.

5°. *The focal point of the congruence corresponding to the variation of* u *is the center of the osculating sphere of the curves* $v = \text{constant}$ *of* S_1; *it is also the center of normal curvature of the curves* $u = \text{constant}$ *of* S_{-1}.

The focal point of the congruence corresponding to the variation of v *is the center of the osculating sphere of the curves* $u = \text{constant}$ *of* S_{-1}; *it is also the center of normal curvature of the curves* $v = \text{constant}$ *of* S_1.

Some of these results can be stated independently of the congruence. Thus we have

THEOREM XIII: *The curves* $u = \text{constant}$ *of* S_{-1} *and* $v = \text{constant}$ *of* S_1 *have common osculating circles at corresponding points. The center of the osculating sphere for either of these curves is the center of normal curvature for the other.*

7. **Geometric characterization of systems** (G). Guichard* has given a converse to one of the preceding results, which may be stated as follows:

Let Σ_1, Σ_2, be surfaces on which the lines of curvature correspond, and at corresponding points let one family of lines of curvature on one surface have the same osculating circles as the other family of lines of curvature on the other surface. Then Σ_1 and Σ_2 are the Laplace transforms of a conjugate system (G).

It is possible to transfer a part of the hypothesis to the conclusion and prove

THEOREM XIV: *Let* Σ_1 *and* Σ_2 *be corresponding conjugate systems, and let one family of curves of* Σ_1 *have at corresponding points the same osculating circles as the other family of* Σ_2. *Then* Σ_1 *and* Σ_2 *are lines of curvature and are the Laplace transforms of a conjugate system* (G).

Since the osculating circles are identical, the osculating planes are also identical. For convenience, suppose that the curves $u = c$ of Σ_1 and $v = k$ of Σ_2 have common osculating planes, and let M be the intersection of the tangent lines of these curves at the points (c, k) of each, called respectively M_{-1} and M_1. Consider the osculating planes corresponding to the values $u = c$, $v = k$, and $u = c$, $v = k + \delta k$. Since they are the osculating planes of neighboring points of $u = c$ on Σ_1 they ultimately intersect in the tangent to that curve. Since they are osculating

* "Sur les cercles osculateurs et les sphères osculatrices aux lignes de courbure d'une surface," Comptes Rendus, 1912, vol. 154, page 678.

planes of $v = k$ and $v = k + \delta k$ at points along the same conjugate curve $u = c$, they are tangent planes to the Laplace transform of Σ_2 at points along the curve $u = c$. The corresponding point of this transform being on the tangent to $v = k$ of Σ_2 at (c, k) is consequently M. M therefore generates one of the Laplace transforms of Σ_2. Similarly it generates the other Laplace transform of Σ_1. Since the lines MM_1 and MM_{-1} are therefore tangent to the curves $u = c$ and $v = k$ respectively at M, it follows from this and foregoing results that the parametric curves on the locus of M form a conjugate system. Hence Σ_1 and Σ_2 are the Laplace transforms of a single conjugate system. The coördinates x, y, z, of the point M therefore satisfy the equation

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{46}$$

where a and b are functions of u and v. The coördinates of M_1 and M_{-1}, the corresponding points of Σ_2 and Σ_1 are given by the equations

$$x_1 = x - \frac{1}{a}\frac{\partial x}{\partial v}, \qquad y_1 = y - \frac{1}{a}\frac{\partial y}{\partial v}, \qquad z_1 = z - \frac{1}{a}\frac{\partial z}{\partial v},$$

and

$$x_{-1} = x - \frac{1}{b}\frac{\partial x}{\partial u}, \qquad y_{-1} = y - \frac{1}{b}\frac{\partial y}{\partial u}, \qquad z_{-1} = z - \frac{1}{b}\frac{\partial z}{\partial u}.$$

Evidently, a necessary and sufficient condition that the curves v = constant of Σ_2 and u = constant of Σ_1 have the same osculating circles is that $MM_1 = MM_{-1}$ and that the curves mentioned have at corresponding points the same polar line. The polar line can be derived from the fact that it is the characteristic of the envelope of normal planes.*

The equation of the normal plane to the curve v = constant on Σ_2 is

$$\Sigma\left(X - x + \frac{1}{a}\frac{\partial x}{\partial v}\right)\frac{\partial x}{\partial v} = 0$$

or

$$\Sigma X\frac{\partial x}{\partial v} - \Sigma x\frac{\partial x}{\partial v} + \frac{G}{a} = 0.$$

For convenience we call the left-hand side of this equation K. The polar line is now given by the equations

$$K = 0, \qquad K_u = 0,$$

where

$$K_u = \Sigma X\left(a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v}\right) - \Sigma x\left(a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v}\right) - F + \frac{\partial G}{\partial u}\frac{1}{a} - \frac{G}{a}\frac{\partial}{\partial u}\log a.$$

* Eisenhart, page 65.

Similarly, the polar line of the curves u = constant on Σ_1 are given by the equations

$$L = 0, \qquad L_v = 0,$$

where

$$L = \Sigma X \frac{\partial x}{\partial u} - \Sigma x \frac{\partial x}{\partial u} + \frac{E}{b}$$

and

$$L_v = \Sigma X\left(a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v}\right) - \Sigma x\left(a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v}\right) - F + \frac{\partial E}{\partial v}\frac{1}{b} - \frac{E}{b}\frac{\partial}{\partial v}\log b.$$

If the polar lines coincide, the planes $K_u = 0$ and $L_v = 0$ must both belong to the pencil $kK + lL = 0$. Now $K_u = 0$ can be written in the from

$$aK + bL - F + \frac{1}{a}\frac{\partial G}{\partial u} - \frac{1}{a}G\frac{\partial}{\partial u}\log a - \frac{a}{b}E - \frac{b}{a}G = 0.$$

If it is in the pencil, we must have

$$F - \frac{1}{a}\frac{\partial G}{\partial u} + \frac{1}{a}G\frac{\partial}{\partial u}\log a + \frac{a}{b}E + \frac{b}{a}G = 0. \tag{47}$$

Similarly, if $L_v = 0$ is in the pencil, we must have

$$F - \frac{1}{b}\frac{\partial E}{\partial v} + \frac{1}{b}E\frac{\partial}{\partial v}\log b + \frac{a}{b}E + \frac{b}{a}G = 0. \tag{48}$$

On account of the identities*

$$aF + bG = \frac{1}{2}\frac{\partial G}{\partial u},$$

$$aE + bF = \frac{1}{2}\frac{\partial E}{\partial v}$$

(47) and (48) become

$$\frac{1}{2a}\frac{\partial G}{\partial u} = \frac{G}{a}\frac{\partial}{\partial u}\log a + \frac{a}{b}E, \tag{49}$$

$$\frac{1}{2b}\frac{\partial E}{\partial v} = \frac{E}{b}\frac{\partial}{\partial v}\log b + \frac{b}{a}G. \tag{50}$$

The condition $MM_1 = MM_{-1}$ gives the relation

$$\frac{E}{b^2} = \frac{G}{a^2}. \tag{51}$$

In consequence of this relation, equations (49) and (50) become

$$\frac{\partial}{\partial u}\log\sqrt{G} = \frac{\partial}{\partial u}\log a + b, \tag{52}$$

* Eisenhart, pp. 153, 154.

$$\frac{\partial}{\partial v}\log\sqrt{E} = \frac{\partial}{\partial v}\log b + a. \tag{53}$$

Hence we can put

$$a = \frac{\partial \varphi}{\partial v}, \qquad b = \frac{\partial \varphi}{\partial u}.$$

Now equations (52) and (53) give

$$G = a^2 e^{2\phi} \cdot V,$$

$$E = b^2 e^{2\phi} \cdot U,$$

where U is a function of u alone, and V of v alone. Equation (51) shows that, without loss of generality, we have $U = V = 1$, and obtain

$$E = \left(\frac{\partial \varphi}{\partial u}\right)^2 e^{2\phi}, \qquad G = \left(\frac{\partial \varphi}{\partial v}\right)^2 e^{2\phi},$$

while equation (46) becomes

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \varphi}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \varphi}{\partial u}\frac{\partial \theta}{\partial v}.$$

The parametric curves on S therefore form a conjugate system (G), and the theorem is proved.

A THEOREM ON LATTICE-POINTS.*

By Aubrey J. Kempner.

Since lattice-point systems are of importance in many branches of mathematics,† there is justification for studying them for their own sake. The problem treated in this paper is also of interest on account of the very elementary methods required and because its complete solution is given by a pointwise discontinuous function, whereas generally such functions are artificially built up in order to illustrate certain peculiarities that real functions may possess.

In a rectangular system of coördinates we consider the lattice-point system x_i, y_j; $x_i = 0, \pm 1, \pm 2, \cdots, y_i = 0, \pm 1, \pm 2, \cdots$. Our problem shall consist of discussing the system of rectilinear paths of finite width and extending to infinity in both directions, which may be laid through our point-system, when points may lie on either boundary-line of the path, but not in the path.

We prove first

Theorem I. *Every straight line in our coördinate-plane belongs to one of the four types*: (*a*) *lines of rational slope containing no points of the system*, (*b*) *lines of rational slope containing an infinite number of our points*, (*c*) *lines of irrational slope containing no points*, (*d*) *lines of irrational slope containing exactly one point.*

Proof. (*a*) Lines satisfying (*a*) obviously exist, for example, $y = \alpha$, or $y = x - \alpha$, α any rational number, but not an integer.

(*b*) is true because a line containing one of our points must necessarily contain a second one when the slope is rational, and a line containing two points (q_1, p_1), (q_2, p_2) contains all points

$$(q_1 + k(q_2 - q_1),\ p_1 + k(p_2 - p_1)), \qquad k = \pm 1, \pm 2, \pm 3, \cdots.$$

Lines parallel to the y-axis are to be counted as of rational slope.

* Presented to the American Mathematical Society, Chicago, April, 1917.

† Such systems are of importance for example in number-theory, in the theory of binary quadratic forms, the theory of algebraic numbers, the theories of elliptic functions and of elliptic modular functions, and, when we consider systems in space, in crystallography. Two of the most valuable studies of plane lattice-point systems are: Klein, Ausgewählte Kapitel der Zahlentheorie, I, II, Hectographierte Vorlesung 1895/6, and Minkowski, Diophantische Approximationen, 1907. Also parts of Minkowski's Geometrie der Zahlen deal with plane lattice-point systems.

In Klein's treatment the lattice-point system is made the basis of the theories of quadratic forms (Pell's equation, etc.), of elliptic functions and of elliptic modular functions.

(*c*) Any line $y = \tan\varphi(x - c)$, $\tan\varphi$ irrational, c rational, but not an integer, does not contain any point of our system.

(*d*) A line of irrational slope clearly does not contain more than one of our points; on the other hand, in case it should contain none, we can draw a line parallel to it through one of our points.

It may be mentioned that Theorem I still holds when we consider the everywhere dense set of points having both coördinates rational; it also holds for the set of points having the algebraic numbers for coördinates, provided we classify our straight lines according to whether their slope has an algebraic or a transcendental value.

THEOREM II.* *Any line of irrational slope has on either side an infinite number of points lying closer to it than any assigned distance.*

Proof. The following theorem is well known and different proofs for it are in existence: "Let α be any real irrational number, then there exist an infinite number of integers p, q such that

$$0 < \left|\frac{p}{q} - \alpha\right| < \frac{1}{q^2}."$$

This theorem states in one respect much more than is needed for the proof of II, while in another respect it does not state quite enough. It will be just as convenient to derive the inequalities which we shall need as to modify the theorem just quoted. We therefore prove first the following

LEMMA. *Let α be any real irrational number, c any real number, and $\epsilon > 0$ arbitrarily small, but fixed. Then there exist four integers p_1, p_2, q_1, q_2 so that*

$$c < q_1\alpha - p_1 < c + \epsilon$$

and

$$c - \epsilon < q_2\alpha - p_2 < c.$$

Proof of lemma. We make use of Minkowski's "zirkulare Anordnung von Intervallen"—circular arrangement of intervals—by means of which he proves the more powerful inequality quoted above (in slightly modified form).†

Consider a circle of radius $1 \cdot 2\pi$, so that the length of the circumference is unity, and, taking any point O on the perimeter as a starting-point, measure along the circumference the lengths α, 2α, 3α, $\cdots$ in inf., going

* In his book mentioned above, Klein gives a much more precise theorem on the degree of approximation in case the line passes through a lattice-point.—By a remarkable geometrical interpretation, Klein's theorem (see p. 17 of his book) places the theory of approximation to irrational numbers by continued fractions in a new light. Compare also MacMillan, A Theorem connected with irrational numbers, Amer. Journal of Math., 1916, v. 38, p. 387. H. J. S. Smith, Collected Mathematical Papers, vol. II, 1894, p. 146, also mentions the same interpretation.

† See: Diophantische Approximationen, p. 4.

around the circle as often as necessary. No two points thus marked can coincide (for *rational* α we should obtain a finite set of equidistant points along the circumference) and, since there are an infinite number of them, they must have at least one point of condensat on. Therefore there are among our points two whose distance measured along the circumference is smaller than ϵ, say equal to ϵ_1, $0 < \epsilon_1 < \epsilon$, and we can find four integers, m_1, m_2, k_1, k_2, some positive and some negative, so that

$$|m_1\alpha - k_1 - (m_2\alpha - k_2)| = |(m_1 - m_2)\alpha - (k_1 - k_2)|$$
$$= |m\alpha - k| = \epsilon_1 < \epsilon,$$

where m, k are again two integers. Then the points

$$|\rho \cdot m \cdot \alpha - \rho \cdot k| = \rho \cdot \epsilon_1, \qquad \rho = 1, 2, 3, \cdots \text{ in inf.},$$

cover the circumference of our circle in such manner that the distance, measured along the circumference, between any two consecutive points is smaller than ϵ, and the point c, measured off on the circumference from the starting-point O, lies in one of these intervals or in an end-point of an interval. This proves our Lemma.

Theorem II is now easily derived: Let $y = m(x - c) = mx - \gamma$, m irrational, c any real number, be the equation of our line. Without loss of generality, m may be assumed positive, as is easily seen. Then $|mq - p - \gamma| = \nu$ is the vertical distance of any point (q, p) from the line and the perpendicular distance δ is smaller than ν. By our Lemma we can find four integers p_1, p_2, q_1, q_2 so that

$$\gamma < q_1 m - p_1 < \gamma + \epsilon$$

and

$$\gamma - \epsilon < q_2 m - p_2 < \gamma,$$

$\epsilon > 0$ arbitrarily assigned, and therefore ν, and all the more δ, an arbitrarily small positive quantity. By choosing ϵ smaller and smaller, we find as many points on either side of our line as we like.

THEOREM III. *When* $\tan \varphi = p/q$, p, q *relatively prime, then the broadest possible path in the direction* φ, *which does not contain any lattice-point in its interior, has the width*

$$d = \frac{1}{\sqrt{p^2 + q^2}}.$$

When $\tan \varphi$ *is irrational, no path of finite width exists in the direction* φ *which does not contain an infinite number of points in the interior.*

Proof. For simplicity of statement, we may assume $0 \leqq \varphi \leqq (\pi/2)$, without real loss of generality. For irrational values of $\tan \varphi$ the proof follows immediately from II. For $\varphi = 0$ and for $\varphi = \pi/2$, our paths are of width unity.

We assume $\tan \varphi = p/q$, p, q relatively prime. If one or both of the border-lines should not pass through one of our lattice-points, we should be able to make the path broader by replacing the actual border-lines by parallel ones. We may, without loss of generality, assume that one of the borders of our path contains the origin; then the distance of any point (q_1, p_1) from this border-line is

$$d = \frac{|qp_1 - pq_1|}{\sqrt{p^2 + q^2}}.$$

Since p, q are relatively prime, it is possible to find integers p_1, q_1 such that $|qp_1 - pq_1| = 1$,* and this is the smallest possible value of the numerator (besides 0, which corresponds to points in the line). This proves III.

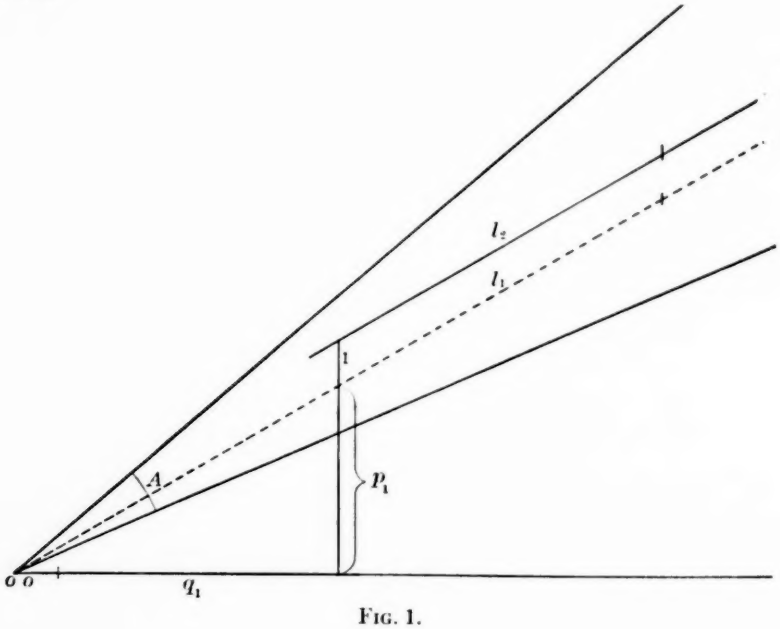

FIG. 1.

THEOREM IV. *Always letting one border (for example the left-hand border) of our paths pass through a fixed lattice-point, and considering only the paths which correspond to a continuous change in direction through any fixed angle, however small, the sum of the widths diverges.*

Proof. Assume the fixed point, through which the left-hand borders of our paths pass, to be the origin, and consider the sector of angle A

* It is easily possible to avoid the diophantine equation and to make the proof entirely elementary. See Theorem V, second proof.

swept over by the left-hand border. We can find in this sector (which we may assume to be in I. Quadrant) at least one line l_1 passing through $(0, 0)$ and a lattice-point (q_1, p_1) (see Fig. 1), and so that also $(q_1, p_1 + 1)$ lies inside of the sector. Then the line l_2 parallel to l_1 and passing through $(q_1, p_1 + 1)$ contains an infinite number of lattice-points $(q_1, p_1 + 1)$, $(2q_1, 2p_1 + 1)$, $(3q_1, 3p_1 + 1)$, $\cdots$ in inf., which all lie in our sector and no two of which are collinear with $(0, 0)$. Consider the set of paths whose left-hand borders pass through $(0, 0)$ and $(\lambda q_1, \lambda p_1 + 1)$, $\lambda = 1, 2, 3, \cdots$. Any pair of numbers $\lambda q_1, \lambda p_1 + 1$ are not necessarily relatively prime, but the effect of using in

$$d = \frac{1}{\sqrt{p^2 + q^2}}$$

for p, q two numbers not relatively prime will be to make our path narrower than necessary. Hence, when we let the left-hand border rotate through a fixed angle, however small, we shall have in

$$\sum_{\lambda=1}^{\infty} \frac{1}{\sqrt{(\lambda q_1)^2 + (\lambda p_1 + 1)^2}}$$

the sum of the widths of only a sub-set of all paths satisfying the conditions of IV, and some terms of the series may be smaller than the width of the corresponding broadest paths. Therefore IV will be proved when we show that

$$\sum_{\lambda=1}^{\infty} \frac{1}{\sqrt{(\lambda q_1)^2 + (\lambda p_1 + 1)^2}}$$

diverges. This follows immediately from

$$\frac{1}{\sqrt{(\lambda q_1)^2 + (\lambda p_1 + 1)^2}} > \frac{1}{\lambda} \cdot \frac{1}{\sqrt{q_1^2 + (p_1 + 1)^2}}.$$

Evidently the sum of the widths still diverges when we consider in any sector only those paths of which the maximum width is smaller than any assigned positive value.

THEOREM V. *Consider a path of maximum width one border of which passes through* $(0, 0)$ *and* (q, p), p, q *relatively prime. The area of the path measured between* $(0, 0)$ *and* (q, p) *is unity.*

This is, as will be seen, exactly the theorem which states that, for doubly-periodic functions (lemniscatic functions corresponding to a *square* lattice-point system), the areas of any two primitive period-parallelograms are equal.

Proof. First proof. The area of the path within the interval considered is equal to a rectangle of sides $\sqrt{p^2 + q^2}$ and $1/\sqrt{p^2 + q^2}$.

Second proof. A geometric proof is indicated by Fig. 2.

We first let the shaded parts of our parallelogram (*abcd*) slide down to the x-axis, and then let those parts which are not already in the square (0, 0), (1, 0), (0, 1), (1, 1) slide along the x-axis in the way indicated by the figure.

To show how the proof applies to the general case, assume for p and q (relatively prime) the rectangle (0, 0), (q, 0), (0, p), (q, p), and draw the diagonal from (0, 0) to (q, p). This diagonal intersects each of the lines $y = k$, ($k = p, p-1, \cdots, 2, 1$); the distances of these points of intersection from the side of the rectangle whose equation is $x = q$ are, respectively,

$$0, \frac{q}{p}, \frac{2q}{p}, \cdots, \frac{(p-1)q}{p}.$$

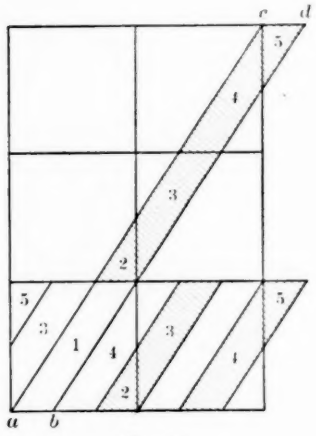

FIG. 2.

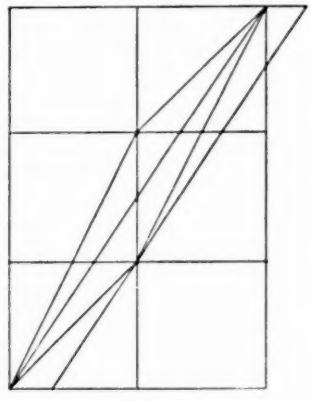

FIG. 3.

Since the numerators of these fractions form a complete residue system modulo p, the horizontal distance of exactly one lattice-point lying in our rectangle to the right of the diagonal must be $1/p$, and the geometrical translations can be carried out as indicated in Fig. 2.

Theorem III can thus be deduced from Fig. 2, when constructed for any given p and q.

Fig. 3 indicates the relation between Theorem V and the theorem on primitive period-parallelograms.

VI. *Discussion of the relation between the width of the path and the angle* φ. We consider all paths of which one border (say the left-hand border) passes through a fixed lattice-point. We may assume this fixed point to be the origin.

The manner in which the width (d) varies with φ will best be described

if we consider d as a function of $\tan \varphi$; $y = f(x)$, writing y for d and x for $\tan \varphi$.

The following properties of $f(x)$ follow immediately from the theorems already proved:

1. $f(x) = 0$ for x irrational.

2. $f\left(\frac{p}{q}\right) = \frac{1}{\sqrt{p^2 + q^2}}$ for p, q relatively prime.

3. $f(0) = f(\infty) = 1$.

4. $f\left(-\frac{p}{q}\right) = f\left(\frac{p}{q}\right)$.

5. $f\left(\frac{p}{q}\right) = f\left(\frac{q}{p}\right)$.

6. $0 \leqq f(x) \leqq 1$ for all values of x.

On account of 4 we may restrict our discussion to positive values of x; on account of 5 we may make the further restriction $0 < x \leqq 1$.

We consider rational values of the argument, *always assuming p, q relatively prime.* We have

$$f\left(\frac{p}{q}\right) = \frac{1}{q} \cdot \frac{1}{\sqrt{1 + \left(\frac{p}{q}\right)^2}}.$$

The second factor on the right side, considered as a function of p/q, does not present any peculiarity. We therefore first investigate the function $F(p/q) = 1/q$, $F(x) = 0$ for x irrational.*

$F(x)$ clearly possesses properties 1 and 6, and we assume $F(0) = 1$. Besides, we have a new property:

7. $F\left(\frac{p}{q}\right) = F\left(1 - \frac{p}{q}\right)$.

* This function is one of the standard examples of pointwise discontinuous integrable functions. Compare for example: Encycl. d. sciences math., II, 1, 2, Montel, Intégration et dérivation, p. 176. Thomae, Bestimmte Integrale, 1908, p. 19. Hobson, Real variables, 1907, gives this function among the exercises on p. 348 as possessing the following properties: (*a*) pointwise discontinuous; (*b*) the Riemann integral (and therefore also the Lebesgue integral) over the interval $0 \cdots 1$ exists: $\int_0^1 F(x)dx = 0$; (*c*) only a finite number of points in the interval $0 \cdots 1$ at which $F(x)$ exceeds an assigned positive number, however small.

See also Hardy, Pure Mathematics, 1908, p. 184, for a related function.

Since, however, a closer investigation of the structure of the function along the lines indicated by our problem was not intended in any of the treatises just referred to, this investigation is carried out in the text.

From the definition of $F(p/q)$ follows:

(*a*) $F(1/n') = 1/n'$ for $n' = 1, 2, 3, 4, \cdots$;

(*b*) $F(2/n'') = 1/n''$ for $n'' = 3, 5, 7, 9, \cdots$, that is, for $n'' > 2$ and n'' relatively prime to 2;

(*c*) $F(3/n''') = 1/n'''$ for $n''' = 4, 5, 7, 8, \cdots$, that is, for $n''' > 3$ and n''' relatively prime to 3; and in general

(*k*) $F(k/n^{(k)}) = 1/n^{(k)}$ for all $n^{(k)}$ such that $n^{(k)} > k$ and $n^{(k)}$ relatively prime to k.

All points of $F(x)$ given by (*a*) lie on the line joining (0, 0) and (1, 1), and they have on this line one point of condensation, at (0, 0). All points given by (*b*) lie on the line joining (0, 0) and (1, $\frac{1}{2}$), and they likewise have one point of condensation, again at (0, 0). In the same way we see that every line passing through (0, 0) and (1, $1/k$), $k = 1, 2, 3, 4, \cdots$, contains an infinite number of points with an only point of condensation at the origin. On account of property 7, every line joining

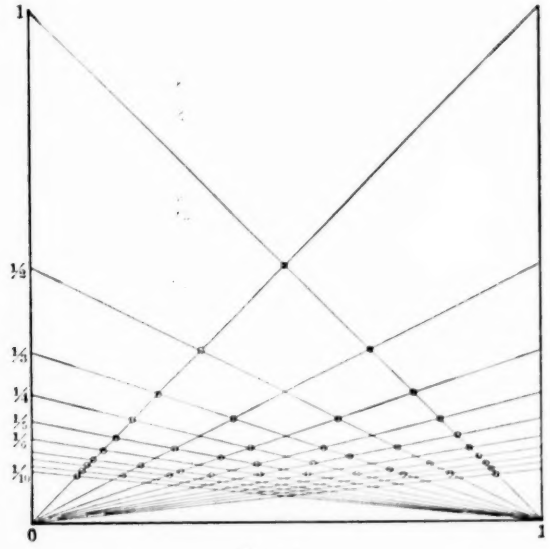

FIG. 4.

(1, 0) and (0, $1/k$) also contains an infinite number of points with an only point of condensation at (1, 0). The points of the locus lying on the second set of lines are the same as the points lying on the first set, and, apart from the everywhere dense set of points lying on the x-axis, all points of our locus are determined as a sub-set of the points of intersection of our two sets of lines. (See Fig. 4.) It is also easy to see how our points are distributed along a set of lines parallel to the x-axis: On

each line $y = 1/n$, $n = 2, 3, 4, 5, \cdots$, we have exactly $\varphi(n)$ points of our locus, where $\varphi(n)$ gives the number of positive integers smaller than n and relatively prime to n, and all points of our locus lie on these lines and on the x-axis (except the two points $f(0) = f(1) = 1$). Hence there are exactly $2 + \sum_{k=2}^{n} \varphi(k)$ points lying on or above the line $y = 1/n$. All except a finite number of points therefore lie in the region $0 < y \leqq \epsilon$, ϵ any positive number. (Compare last footnote.)

It is now easy to pass from $F(x)$ to $f(x)$, since we must only multiply every ordinate of $F(p/q)$ by $1/\sqrt{1 + (p/q)^2}$.

Since $1/\sqrt{1 + x^2}$ is a monotone decreasing function in the interval $0 \cdots 1$, starting with the functional value 1 and ending with $\frac{1}{2}\sqrt{2}$, the effect of the multiplication is obvious.

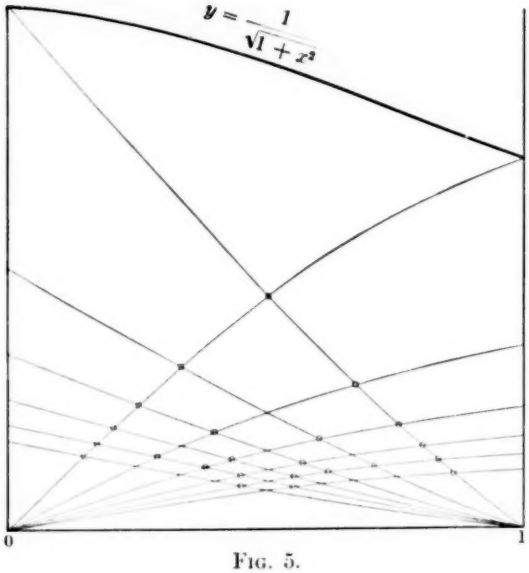

FIG. 5.

The set of straight lines starting from (0, 0) will be transformed into a set of curves through (0, 0), each one of which is concave downward and lies entirely underneath the corresponding straight line; while the set of lines starting from (1, 0) will be transformed into a set of curves each one of which passes through (1, 0) and also through the point of intersection of the corresponding straight line with the y-axis, while for intermediate values of x the (concave upward) curve lies entirely below the line. The points of intersection of our two sets of curves, corresponding to those points of intersection in Fig. 4 which are points of $y = F(x)$, yield points of $y = f(x)$, and exactly all points of $f(x)$ except the everywhere dense

set of irrational points lying on the x-axis (see Fig. 5). Like $F(z)$, our function is pointwise discontinuous, integrable from 0 to 1, with

$$\int_0^1 f(x)dx = 0,$$

and all but a finite number of points lie within any band $0 < y \leqq \epsilon$, ϵ any given positive number, for $0 < x < 1$.

The broadest paths in our system are approximately realized, and are frequently clearly visible, in a well-planted corn field.

Without important modifications, the investigation may be carried over to rectangular and parallelogram arrangements of points, and therefore in particular to the case when our points are vertices of a system of equilateral triangles filling the plane. In generalizing into space-systems, it will be natural to consider, corresponding to the paths in the plane, the space between parallel planes which may be laid through the system so that no lattice-points lie between the planes.

URBANA, ILL.

ON CONTINUOUS REPRESENTATIONS OF A SQUARE UPON ITSELF.

By H. L. Smith.

In a recent paper under the above title, H. Tietze* has shown that every one-to-one, continuous representation of a square upon itself which preserves sense is a deformation; that is, it may be regarded as a member of a continuous, one-parameter family of such representations which contains the identity. The greater part of Tietze's paper, which occupies 57 pages, is concerned with the proof of a fundamental lemma, which is done by means of a process of successive approximations. In the present paper that lemma is proved in a simple manner by using the theorem of Schoenflies† that two closed, simply connected regions whose boundaries are simple closed curves can be represented in a one-to-one, continuous fashion upon each other so that the correspondence so determined between their boundaries is a preassigned one-to-one, continuous one.‡

For the sake of brevity and concreteness, use is made in the present paper of metric devices, but it will be readily seen that such devices are not essential. The proof admits also of generalization to n-dimensions, provided that such is the case for the Schoenflies theorem.

§ 1. Preliminary definitions and lemmas.

For the sake of simplicity the theorems of this paper will for the most part be stated for the square

$$\mathfrak{S}: \qquad |x + y| + |x - y| \leqq 2;$$

it will be apparent that they may be generalized to the case of any point set that is in one-to-one, continuous correspondence with the square $\mathfrak{S}$.

A one-to-one, continuous representation R of the square $\mathfrak{S}$ upon itself, which is denoted by the equations

$$R: \qquad \xi = f(x, y), \qquad \eta = g(x, y),$$

is said to be a *deformation* provided there exists a pair of functions $\varphi(x,$

* Rendiconti di Palermo, vol. 38 (1914), p. 247.

† Bericht über die Entwicklung der Lehre von den Punktmannigfaltigkeiten, Part II, p. 109.

‡ Since the present paper was written, the writer has found a paper of Tietze's (Sitzungsberichte der Akademie der Wissenschaften in Wien (1913), p. 1653) in which among others he deduces the Schoenflies theorem. It does not seem to have occurred to him to deduce the Tietze theorem from the theorem of Schoenflies.

$y, t)$, $\psi(x, y, t)$, where, in particular, the range for t may be taken as $0 \leqq t \leqq 1$, and such that the equations

$$\xi = \varphi(x, y, t), \qquad \eta = \psi(x, y, t)$$

define for each t a one-to-one, continuous representation of $\mathfrak{S}$ upon itself, the one corresponding to $t = 0$ being the identity and the one corresponding to $t = 1$ being R, and where, finally, φ and ψ are continuous in (x, y) and t separately. If φ and ψ are continuous in x, y, t simultaneously, then R is a deformation *in the strong sense*.

It is easily shown that the product of two deformations (in the strong sense) of $\mathfrak{S}$ is again a deformation (in the strong sense) of $\mathfrak{S}$.

We also assume the Tietze theorem for the one dimensional cases.*

LEMMA 1. *If j_* are two simple curves joining the points A_*: $(\pm 1, 0)$, having no points in common except their end points A_*, and lying within $\mathfrak{S}$ with the exception of their end points, then there is a deformation of $\mathfrak{S}$ into itself that leaves the boundary of $\mathfrak{S}$ pointwise invariant and carries j_+ into j_- in such a way as to preserve a preassigned one-to-one, continuous correspondence T between j_*.*

We assume that the notations for the curves j_* are so chosen that the region bounded by the simple closed curve made up of the points of j_+ together with the points of the boundary of $\mathfrak{S}$ whose ordinates are greater than zero contains no point of j_-. Then there exists, by the theorem of Schoenflies above referred to, a one-to-one, continuous representation π of $\mathfrak{S}$ upon itself such that $\pi(j_*) = k_*$, where k_* are defined by the equations†

$$k_*: \qquad y = \pm \tfrac{1}{2} \operatorname{sm}(|1 + x|, |1 - x|) = \pm \omega(x),$$

and such that the boundary of $\mathfrak{S}$ is left pointwise invariant. The representation π may (and will) be so chosen that if P_* are points of j_* which correspond according to T, then $\pi(P_*)$ have the same abscissa. In the above sm is an abbreviation for "the smaller."

There exists a deformation in the strong sense of the square $\mathfrak{S}$ which leaves the boundary pointwise invariant and carries k_+ into k_- in such a way that the image of each point has the same abscissa as the point itself. For, if we denote by sgn z, 1, 0, or -1 according as z is positive, zero, or negative, such a deformation is given by the equations‡

* Tietze, loc. cit., p. 249.

† k_+ is clearly the broken line gotten by joining the point pairs $(-1, 0)$, $(0, 1/2)$ and $(0, 1/2)$, $(1, 0)$ by line segments.

‡ This is clearly an up-and-down stretching of the square.

$$D: \quad \begin{cases} \xi = x \\ \eta = \tau[x, y, t, \kappa(y - t\omega(x))] \end{cases} \qquad 0 \leqq t \leqq 1,$$

where

$$\begin{cases} \kappa(p) = \operatorname{sgn} p + 1 - \operatorname{sgn} p^2, \\ \tau(x, y, t, q) = q\left[1 - (1 - yq)\dfrac{1 + tq\omega(x)}{1 - tq\omega(x)}\right]. \end{cases}$$

We can now prove the lemma. For $\pi^{-1}D\pi$ is a deformation which has the properties there described.

LEMMA 2. *There exists a simple curve joining A_{-}, lying within $\mathfrak{S}$ except for its end points A_{-}, and having no points in common with j_{+} except its end points A_{-}.**

Let π denote a one-to-one, continuous representation of $\mathfrak{S}$ upon the circle whose equation in polar coördinates is $\rho = 1$, such that $A_{\pm}$ are each invariant. Let J denote the image of the points of j_{+}. Let $\lambda(t)$ be the least upper bound of values of ρ for points (ρ, t), $0 \leqq t \leqq \pi$, which are projections on the line $\theta = t$ of points of J. Then the equations

$$C: \quad \begin{cases} \rho = \frac{1}{2}[1 + \lambda(t)], \\ \theta = t, \end{cases} \qquad 0 \leqq t \leqq \pi$$

define a simple curve† lying in the circle $\rho = 1$ with the exception of its end points A_{-}, and having no points in common with J except A_{-}. Then $\pi^{-1}C$ is the curve of the lemma.‡

FUNDAMENTAL LEMMA. *If T is a one-to-one, continuous representation of $\mathfrak{S}$ upon itself which leaves the boundary pointwise invariant, then there exists a deformation D in the strong sense of $\mathfrak{S}$ such that DT^{-1} leaves pointwise invariant the boundary of $\mathfrak{S}$ and the segment l joining A_{-}.*

Set $T^{-1}(l) = j$. Then by Lemma 2 there is a curve k joining A_{+}, lying within $\mathfrak{S}$ except for the points A_{-}, and having no points in common with j, l except A_{-}. By Lemma 1 if c denotes any correspondence from l to k, there exists a deformation in the strong sense D_1 which carries j into k in such a way as to preserve the correspondence cT on j to k, the boundary of $\mathfrak{S}$ remaining invariant. Further there exists a similar deformation D_2 which carries k into l in such a way as to preserve the correspondence c^{-1} on k to l, the boundary remaining invariant. Then $D = D_2D_1$ is the deformation required in the lemma.

* Professor R. L. Moore has indicated to me how this lemma may be easily proved on the basis of Theorems 44 and 40 of his paper, On the Foundations of Plane Analysis Situs (Transactions of the American Mathematical Society (1916), p. 155). Such a proof is not completely metric.

† This is a consequence of the theorem that if a function $f(xt)$ is uniformly continuous in x and t, then the least upper bound of $f(xt)$ with respect to t is a continuous function of x.

‡ The proof from here on is not essentially different from that of Tietze.

GENERALIZED LEMMA. *If l and m are two segments which divide a square into four congruent squares, and if T is a one-to-one, continuous representation of that square upon itself which leaves the boundary pointwise invariant, then there exists a deformation Δ in the strong sense of the square such that ΔT^{-1} leaves the boundary of the square, l, and m pointwise invariant.*

By the above fundamental lemma there is a defomation D of the square which is such that DT^{-1} leaves pointwise invariant the boundary and l. Similarly* there is a deformation d_+ of one of the two rectangles into which the square is divided by l, which exists in the strong sense and is such that D_+ (DT^{-1}), leaves pointwise invariant the boundary of the rectangle and the part of m lying within the rectangle. A similar deformation exists for the other rectangle; call it d_-. Combining the ranges of d_*, we obtain a deformation d over the entire square such that $dD = \Delta$ will serve as the required deformation.

§ 2. Proof of the theorem.

THEOREM. *A one-to-one, continuous representation R of a square $\mathfrak{S}$ upon itself which preserves sense on the boundary is a deformation in the strong sense.*

Suppose R leaves the boundary pointwise invariant. Then consider the set of lines

L_n: $\qquad x = \pm .a_1 \cdots a_n, \qquad y = \pm .b_1 \cdots b_n,$

where the right members of the equations are decimals expressed in the scale of 2, and where L_0 is defined specially to be the pair of lines $x = 0$, $y = 0$.

There exists for each n a deformation Δ_n of $\mathfrak{S}$ in the strong sense which is such that $\Delta_n R^{-1}$ leaves invariant all points of $\mathfrak{S}$ on the boundary and on the lines L_n. This is clearly true for $n = 0$ by the generalized lemma. To prove it for $n + 1$ on the hypothesis that it holds for n, we have only to apply the generalized lemma to each of the squares into which L_n divides $\mathfrak{S}$; the deformations so found when united form a single deformation over the entire square $\mathfrak{S}$, this deformation multiplied on the right by Δ_n gives Δ_{n+1}.

Since the deformation Δ_n is identical with R on the boundary of $\mathfrak{S}$ and on L_n, the limit of Δ_n is R, and a simple construction shows that this limit is a deformation, and further, one in the strong sense.†

The General Case. Suppose T is a one-to-one, continuous representation of $\mathfrak{S}$ upon the circle whose equation in polar coördinates is $\rho = 1$. Set $C = TRT^{-1}$, and if we denote by $\theta' = \nu(\theta)$ the one-to-one,

* In applying the fundamental lemma take the T of that lemma as TD^{-1}.

† Cf. Tietze, loc. cit., p. 253.

continuous representation of the peripherie of the circle upon itself which is defined by C, and if c is the one-to-one, continuous representation of the circle upon itself given by the equations

$$c: \qquad \rho' = \rho, \qquad \theta' = \nu(\theta),$$

then the equation

$$R = (T^{-1}cT)(T^{-1}c^{-1}CT)$$

expresses R as the product of two one-to-one, continuous representations of $\mathfrak{S}$ upon itself, the second of which leaves pointwise invariant the boundary of $\mathfrak{S}$, and is therefore a deformation. That the first one is also a deformation follows readily from the fact that by the Tietze theorem for the one-dimensional case, C, and therefore c, is a deformation. Hence R is a deformation, and the theorem is proved.

PRINCETON UNIVERSITY,
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ROOTS AND SINGULAR POINTS OF SEMI-ANALYTIC FUNCTIONS.

By Dunham Jackson.

Various theorems about the behavior of a function of several complex variables in the neighborhood of a point have analogous theorems corresponding to them for the case of a function which is analytic with respect to a particular one of its arguments, but merely continuous with regard to the whole set of variables, a function of the class which Bôcher has called semi-analytic.* It is the purpose of this note to point out some of the cases in which the analogy holds, and a simple case in which it is not preserved unimpaired. The method of treatment is closely related to that of an earlier paper by the author, on analytic functions.†

1. Weierstrass's factor theorem.

In proving Weierstrass's preparation theorem by means of Cauchy's integral formula,‡ the assumed analytic character of the function in question with regard to the totality of its arguments is used in such a way as to suggest that if a part of this analytic character were sacrificed in the hypothesis, nothing worse would result than a precisely corresponding weakening of the conclusion. By way of illustration, let us consider a function $f(x, y)$, defined throughout a region

$$|x| \leqq h, \qquad |y| \leqq h, \tag{1}$$

surrounding the origin, the variable x being complex, and the variable y, for simplicity, being taken as real. Let it be assumed that if y is given any fixed value in the interval under consideration, the function f, with respect to its dependence on x alone, is analytic throughout the circle $|x| \leqq h$, while $f(x, y)$ is continuous in both arguments throughout the region (1). The function $f(x, 0)$ is assumed not to vanish identically, but to have a root, of order m let us say, at the point $x = 0$. It will be seen that $f(x, y)$ admits of factorization in the neighborhood of the origin, in accordance with a formula exactly like that which is ordinarily obtained for analytic functions.

* Bôcher, On semi-analytic functions of two variables, *Annals of Mathematics*, vol. 12 (1910–11), pp. 18–26.

† D. Jackson, Non-essential singularities of functions of several complex variables, *Annals of Mathematics*, vol. 17 (1915–16), pp. 172–179. This paper will be referred to simply by the letter A.

‡ Cf., e. g., Picard, Traité d'analyse, vol. 2, p. 244 of the first edition, 264 of the second edition.

Let $r_1 < h$ be a positive number such that $f(x, 0)$ does not vanish for $0 < |x| \leqq r_1$. The circle $|x| = r_1$, $y = 0$, is interior to a region of the space of the variables x, y, throughout which f is continuous in both variables and different from zero. The property of continuity extends to the derivative $f_x(x, y)$, as was pointed out by Bôcher.* Form the contour integral

$$\int_C \frac{f_x(x, y)}{f(x, y)} dx,$$

with the circle $|x| = r_1$ as path of integration. Since the integrand is continuous in both arguments, the value of the integral is a continuous function of y in the neighborhood of the value $y = 0$. On the other hand, it is recognized that for each particular value of y the integral gives the number of roots of the function $f(x, y)$ inside the circle $|x| \leqq r_1$, multiplied by $2\pi i$. The term "number of roots" is used for brevity to mean "sum of the orders of the roots." Since this number is always a positive integer and depends continuously on y, it must be constantly equal to m.

That is, it is possible to write down, for each particular value of y within a sufficiently restricted interval about the origin, a polynomial of the mth degree,

$$\psi(x, y) = x^m - p_1(y)x^{m-1} + p_2(y)x^{m-2} + \cdots \pm p_m(y), \tag{2}$$

whose roots coincide in location and multiplicity with those of $f(x, y)$. Nothing is said for the moment with regard to the character of the functions $p_r(y)$; the statement at present is merely the obvious one that such functions exist. The quotient

$$\varphi(x, y) = \frac{f(x, y)}{\psi(x, y)},$$

for each particular value of y, has only removable singularities as a function of x, and, if suitably defined at the points where these singularities occur, is analytic in x and different from zero throughout the circle $|x| \leqq r_1$. For x in this circle, then, and for y in a suitable interval $|y| \leqq r_2$, f has the representation

$$f(x, y) = \psi(x, y)\varphi(x, y), \tag{3}$$

where ψ has the form (2), and φ is everywhere different from zero. So much is immediate; it remains to discuss the continuity of the functions p_r and φ.

* Loc. cit., p. 20; the fact that the derivative is continuous in both variables may be recognized by expressing it as a contour integral, and making use of the continuity of the integrand.

Let C denote the same circle as before. The expression

$$\frac{1}{2\pi i}\int_C \frac{x^k f_x(x, y)}{f(x, y)}\,dx,$$

where k is any positive integer, gives the sum of the kth powers of the values of x in the circle for which $f(x, y)$ vanishes, each repeated according to the multiplicity of the corresponding root—for brevity, the sum of the kth powers of the roots. Each integral is a continuous function of y, because of the continuity of the integrand in both arguments. Since the sums of powers are continuous functions of y, the same thing is true of the elementary symmetric functions of the roots, $p_1(y), \cdots, p_m(y)$, which form the coefficients in (2).

Consider now the function φ, supposed to have been defined at its points of removable discontinuity so as to be analytic in x without exception. Its value can be written in the form

$$\varphi(x, y) = \frac{1}{2\pi i}\int_C \frac{\varphi(t, y)}{t - x}\,dt.$$

The integrand is continuous in t, x, and y, on the path of integration, if x is inside the circle and y is sufficiently small. Hence φ, obviously continuous in both its arguments except at the points where singularities were removed, is continuous there also.

The conclusion may be stated in the following form:

THEOREM I. *Let $f(x, y)$ be a function of the complex variable x and the real variable y, defined throughout a region containing the origin, analytic in x, and continuous in both variables together. Let $f(x, 0)$ vanish for $x = 0$, without vanishing identically. Then, throughout a sufficiently restricted neighborhood of the origin, $f(x, y)$ can be factored in accordance with equation (3), where $\varphi(x, y)$ is analytic in x, continuous in both variables, and everywhere different from zero, and $\psi(x, y)$ has the form (2), the coefficients $p_r(y)$ being continuous functions of y which vanish for $y = 0$.*

The validity of the very last clause is an immediate consequence of the definition of ψ.

A considerable part of the substance of this theorem, showing as it does that the roots of f depend continuously on y, must be regarded as well known; but the author does not remember having seen exactly the present formulation of it. A similar remark may apply to some of the later theorems.

Interpreted as a theorem on implicit functions, the assertion is, for $m = 1$, that if $f(x, y)$ is analytic in x and continuous in x and y in the neighborhood of the origin, if, further, $f(0, 0) = 0$, but $f_x(0, 0) \neq 0$, then

the equation

$$f(x, y) = 0$$

defines x as a single-valued continuous function of y for sufficiently small values of y. It will be observed that there is no explicit hypothesis as to continuity of $f_x(x, y)$. In this connection, however, it must be taken into account that f is assumed to be continuous in both variables throughout a region in which x varies over a two-dimensional part of the complex plane; examples can readily be constructed to show that the theorem is not applicable when the continuity is granted merely for real values* of x and the assumed range of values of y. The distinction is analogous to that which is brought out by a Fourier's series such as

$$\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots,$$

which is a uniformly convergent series of analytic functions, in a certain sense, but does not represent an analytic function.† It should be remarked, furthermore, that the continuity of $f_x(x, y)$ is implicitly provided for in the hypothesis as stated for a complex region, by virtue of the theorem of Bôcher already referred to.

It is obvious that Theorem I is capable of extension, with only the most superficial formal changes in statement and proof, to the case of a function $f(x, y_1, \cdots, y_n)$, with any number of real parameters. The assumed reality of the parameters is no restriction then, since a complex variable may be regarded as a pair of real variables.

A somewhat less trivial extension is obtained by including in the hypothesis an assumption that $f(x, y)$ can be differentiated q times with regard to y, and that the qth derivative is a continuous function of x and y together. Under these circumstances, the integrals concerned, whether in the proof of Bôcher's theorem about the derivative of f with regard to x, or in the reasoning of the present paper, can be differentiated q times with regard to y under the sign of integration, and the result of the differentiation will be a continuous function in each case. To give all the details of the reasoning would be a decidedly long process, but the individual steps are perfectly simple. The conclusion is as follows:

* E. g., let

$$f(x, y) = x \text{ for } y \leqq 0, \qquad f(x, y) = x + \frac{x}{y \log y} e^{-\frac{x^2}{y^2}} \text{ for } y > 0.$$

For the continuity, notice that

$$\left| \frac{x}{y \log y} e^{-\frac{x^2}{y^2}} \right| \leqq \frac{\gamma}{\log y},$$

if γ is the maximum of te^{-t^2} (all variables regarded as real). The equation $f(x, y) = 0$ gives

$$x = 0 \text{ for } y \leqq 0, \qquad x = 0 \text{ or } \pm y\sqrt{\log (1/y) - \log \log (1/y)} \text{ for } y > 0.$$

† Cf., e. g., Byerly, Fourier's series and spherical harmonics, pp. 39, 40.

THEOREM Ia. *If $f(x, y)$ in Theorem I has a qth derivative with regard to y, which is continuous in both variables, then the functions $p_r(y)$ and $\varphi(x, y)$ also will have continuous qth derivatives with regard to y, the qth derivative of φ being continuous in both variables.*

This result also can be extended so as to take account of any derivative, of any order, with regard to any number of parameters* $y_1, \cdots, y_n$.

2. Singularity of the first kind.

If a function $f(x, y)$ is analytic in x, except for poles, for each particular value of y, the distribution of the poles is said to be continuous at a point (a, b), if the following condition is satisfied:† There exists a positive number r_1, as small as may be desired, with a corresponding positive number r_2, such that for $|y - b| \leqq r_2$ there are no poles on the boundary of the circle $|x - a| \leqq r_1$, and the sum of the orders of the poles inside the circle is constant. With regard to such functions, this assertion may be made:

THEOREM II. *Let $f(x, y)$ be a function of the complex variable x and the real variable y, having the following properties in the neighborhood of the origin:*

(a) If y is held fast, $f(x, y)$ is an analytic function of x, except for poles.

(b) The distribution of these poles is continuous at the origin.

(c) Except at the points corresponding to poles, f is continuous in both variables together.

Then f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points specified above, in the form

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)},$$

where φ and ψ are analytic in x and continuous in both variables throughout the neighborhood in question, and φ does not vanish there.

The denominator may be taken as a polynomial in x, with coefficients which are continuous functions of y.

The proof corresponds precisely to that of Theorem I of the paper A, and moreover closely resembles the proof of Theorem I of the present paper, so that it is unnecessary to give it here. The theorem can be extended at once to the case of several parameters, or to the case of func-

* It is readily seen that if $f(x, y)$ and its qth derivative with regard to y are continuous in both variables, the same is true of the intermediate derivatives with regard to y. A corresponding theorem does not hold for cross derivatives with regard to several parameters; but if the continuity of the intermediate derivatives is postulated explicitly, the reasoning can be carried through without difficulty.

† See the paper A, pp. 172–73.

tions having continuous derivatives of any order with regard to the parameter or parameters.

3. Singularity of the second kind.

When the attempt is made to adapt Theorem II of the paper *A* to functions of the sort that we are considering here, essential differences between analytic and semi-analytic functions begin to manifest themselves. A theorem will be obtained, to be sure, with comparatively slight changes in conclusion and proof; but the necessity of the changes that are made must not be overlooked. The hypothesis will be as follows; the conclusion will be stated after an exposition of the reasoning by which it is established:

HYPOTHESIS OF THEOREM III. *Let $f(x, y)$ be a function of the complex variable x and the real variable y, having the following properties in the neighborhood of the origin:*

(a) If y is held fast, $f(x, y)$ is an analytic function of x, except at most for poles.

(b) The distribution of these poles is continuous at every point except the origin itself.

(c) Except at the origin and at the points corresponding to poles, f is continuous in x and y together.

The first step of the proof, in the analytic case, consists in showing that there is a region $|x| \leqq \rho_1$, $|y| \leqq \rho_2$, such that if y_0 is any particular point subject to the condition $0 < |y_0| \leqq \rho_2$, the number of poles of $f(x, y)$ in the circle $|x| \leqq \rho_1$ remains constant for all values of y in a sufficiently restricted neighborhood of the point y_0. So much is readily proved, in the same way, in the present case also. Then the inference is drawn, by means of the Heine-Borel theorem, that the number of poles in the circle is constant throughout the whole region $0 < |y| \leqq \rho_2$. It is at this point that a difference appears. For when the parameter y is restricted to real values, the condition $0 < |y| \leqq \rho_2$ does not define a single connected region, but a pair of distinct intervals, and there is no way of passing from one to the other. Taking note of the fact that the number of poles in the circle of the x-plane is constant for $0 < y \leqq \rho_2$ and constant also for $0 > y \geqq -\rho_2$, we will show that these two constant numbers may perfectly well be different.

Consider, for example, a function $f(x, y)$, defined as follows:

$$f(x, y) = \frac{x + y}{x - y} \text{ for } y > 0; \qquad f(x, y) = 1 \text{ for } y \leqq 0.$$

It satisfies all the conditions of the hypothesis, but has not the same

number of poles for positive and for negative values of y. If this function is expressed in the form $f = \varphi/\psi$, where φ and ψ are analytic in x and continuous in both variables, then, by Theorem I, the number of values of x for which $\varphi = f\psi$ vanishes must remain constant when y varies slightly in either direction from the value $y = 0$. There must surely be a root for $y > 0$, so there must be one also for $y < 0$. When $y < 0$, however, $f = 1$, $\psi = \varphi$, and ψ also must have a root; and at these points where φ and ψ vanish simultaneously, the representation can not be said strictly to hold, although f itself is continuous.* As a matter of fact, it is obvious that f can be represented as a quotient φ/ψ, by setting

$$\varphi = x + y \text{ for } y \geqq 0, \qquad \varphi = x \text{ for } y \leqq 0,$$

$$\psi = x - y \text{ for } y \geqq 0, \qquad \psi = x \text{ for } y \leqq 0.$$

It will appear that this simple example is typical of what can, as well as of what can not, be expected to hold in general.

Let us return to the consideration of a general function $f(x, y)$ satisfying the hypothesis. The condition on the number ρ_1, mentioned above, is merely† that $f(x, 0)$ be analytic for $|x| = \rho_1$. It may be supposed further that ρ_1 is chosen so small that $f(x, 0)$ has no pole nor root‡ for $0 < x \leqq \rho_1$. We have seen that the number of poles of $f(x, y)$ in the circle $0 \leqq x \leqq \rho_1$ is constant, let us say equal to m, for $0 < y \leqq \rho_2$. For the present, let y be restricted to this interval, closed, on occasion, by the value $y = 0$; similar reasoning will of course apply to the interval $0 \geqq y \geqq -\rho_2$, considered by itself. The circle in the x-plane will be denoted by R_1.

In the continuation of the proof for the analytic case, it is shown that $p_1(y), \cdots, p_m(y)$, the elementary symmetric functions of the poles in R_1, are analytic functions of y for $0 < |y| \leqq \rho_2$. The proof is given by means of contour integrals extended around small circles interior to R_1. It is clear from the definition of the functions $p_r(y)$ that they remain finite as y approaches 0, and so it is inferred that if suitably defined for $y = 0$ they are analytic there. The part of the reasoning which deals with the contour integrals can be adapted to the present hypothesis, and shows that each $p_r(y)$ is continuous for $0 < y \leqq \rho_2$. But even though $p_r(y)$

* A similar remark would apply to a representation in which φ and ψ vanish identically for $y = 0$, although Theorem I could not be used.

† It is naturally assumed throughout that the variables are restricted to the region for which the hypothesis holds.

‡ This is impossible in the special case, not excluded by the hypothesis, that $f(x, 0)$ vanishes identically. But the single passage in the subsequent reasoning where use is made of the assumption that there is no root for $0 < x \leqq \rho_1$, becomes entirely superfluous if $f(x, 0)$ does not have a pole at the origin.

does remain finite as y approaches zero, there is of course no general theorem for functions of a real variable which makes it possible to extend the continuity to the point $y = 0$. It is to be shown by independent reasoning that $\lim_{y=0} p_r(y) = 0$. The proof is almost immediate, as soon as its necessity has been pointed out.

Let ϵ be an arbitrarily small positive quantity. By the condition imposed on ρ_1, $f(x, 0)$ is analytic for $\epsilon \leqq x \leqq \rho_1$. It follows from the hypothesis of continuity in the distribution of the poles, that $f(x, y)$ is analytic in x throughout the same region, if y is sufficiently small. That is, for small positive values of y, all m of the poles which $f(x, y)$ must have in R_1 are within the circle $|x| \leqq \epsilon$. This means that

$$|p_1(y)| < m\epsilon, \qquad |p_2(y)| < \frac{m(m-1)\epsilon^2}{2},$$

and so on. If the functions $p_r(y)$ are set equal to 0 for $y = 0$, they are continuous for $0 \leqq y \leqq \rho_2$.

Let

$$\psi(x, y) = x^m - p_1(y)x^{m-1} + \cdots \pm p_m(y), \qquad \varphi(x, y) = f(x, y)\psi(x, y).$$

In the case of analytic functions, Cauchy's integral formula is used to prove that after the elimination of removable singularities, φ is analytic in both variables, except at the single point (0, 0); and since an isolated singularity of an analytic function of two variables must be removable, φ can be made analytic without exception. Once more the extension to semi-analytic functions requires a little care. In the application of Cauchy's integral formula there is no difficulty; it is recognized that φ is continuous in x and y together, as well as analytic in x, if removable singularities are disposed of and the point (0, 0) is left out of account. But for this last point special considerations are required.

The function $f(x, 0)$ may be analytic at the origin, or it may have a pole there. In the latter case, the order of the pole will not in general be equal to m (can not be equal to m, if the distribution of the poles is actually discontinuous at the origin). The essential thing is, that a pole of order higher than m is impossible. Consider the integral of f_x/f, extended around the circle $|x| = \rho_1$. It is a continuous function of y throughout the neighborhood of the value $y = 0$. It gives the difference between the number of poles and the number of roots inside the circle, each counted according to its multiplicity. Therefore, being continuous, it is constant. For $y > 0$, the number of poles is m, the number of roots is essentially positive or zero, and the difference can not exceed m. For $y = 0$, there are no roots at all, and the difference, still having the same

value as before, is now simply the number of poles, in other words, the order of the single pole that remains. Thus the last difficulty is disposed of. The function $\psi(x, 0)$ reduces to the single term x^m. The product φ has at most a removable singularity at the origin. When this is closed up, φ is analytic in x, and, by an application of Cauchy's integral formula once more, continuous in x and y together.

The essential conclusion is, that under the conditions of the hypothesis, f can be expressed in the form φ/ψ, where φ and ψ are semi-analytic for $0 \leqq |x| \leqq \rho_1$, $0 \leqq y \leqq \rho_2$, the denominator vanishing only at singular points of f.

The only unexpected feature here, if it may be called unexpected, is the restriction of y to a one-sided neighborhood of the origin. For the interval $0 \geqq y \geqq -\rho_2$, as has already been indicated, a similar conclusion can be drawn. Let m_1 be the number of poles for $y < 0$, the degree of the corresponding polynomial ψ. If it happens that $m = m_1$, both polynomials ψ join on continuously to the function x^m for $y = 0$. In this case, ψ is semi-analytic, and φ can be made semi-analytic, throughout the whole neighborhood $0 \leqq |x| \leqq \rho_1$, $0 \leqq |y| \leqq \rho_2$. Otherwise, suppose $m > m_1$. If

$$\psi_1(x, y) = \psi(x, y) \text{ for } y > 0, \qquad \psi_1(x, y) = x^{m-m_1}\psi(x, y) \text{ for } y < 0,$$
$$\psi_1(x, 0) = x^m,$$

then ψ_1 is semi-analytic throughout the neighborhood of the origin, $\varphi_1 = f\psi_1$ is semi-analytic except for removable singularities, and f is equal to φ_1/ψ_1, in which representation the denominator may vanish at points which are not singular points of f. The analogy between analytic and semi-analytic functions is perhaps upheld as far as possible in the following enunciation:

THEOREM III. *Under the hypothesis already stated, f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points specified, and possibly one other point for each positive or each negative value of y, in the form*

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)},$$

where φ and ψ are analytic in x and continuous in both variables throughout the neighborhood in question.

It may be remarked, without details of proof, that the restrictive phrase " and possibly one other point . . . " can be dispensed with, even when $m \neq m_1$, provided only that neither m nor m_1 is zero; for the extraneous factor in one of the functions ψ can be so chosen as not to

introduce any new root, but merely to increase the multiplicity of one already present.

When it is attempted to extend this theorem, as earlier theorems were extended, by supposing the functions to possess continuous derivatives with regard to y, the analogy fails still further. Consider, for example, the function*

$$f(x, y) = \frac{y}{x - \sqrt{y}}, \qquad f(0, 0) = 0.$$

When y is held fast, it is an analytic function of x, except for a single pole when $y \neq 0$. The situation of the pole varies continuously with y, except for $y = 0$, where it ceases to exist. The function is continuous in x and y together, and has a derivative with regard to y,

$$f_y = \frac{x - \frac{1}{2}\sqrt{y}}{(x - \sqrt{y})^2},$$

which is continuous in both variables, at all points except those where the denominator vanishes. Nevertheless, the function $p_1(y) = \sqrt{y}$ does not have a continuous derivative, even for the one-sided neighborhood $0 \leqq y \leqq \delta$. A representation as a quotient of two functions with continuous derivatives can be obtained by setting $\varphi = y^2$, $\psi = xy - y\sqrt{y}$.

It is obvious that the discussion could be carried further, but enough has been said to illustrate the possibilities of the theory and its limitations. In particular, it may be remarked that a semi-analytic function of one complex and two real variables, unlike an analytic function of three complex variables, can have a singularity of the sort considered in Theorem III, with no other singularities of the same sort in its neighborhood, e. g.,

$$f(x, y, z) = \frac{x^2 + y^2 + z^2}{x^2 - y^2 - z^2}, \qquad f(0, 0, 0) = 1 \qquad (y \text{ and } z \text{ real}).$$

The neighborhood of the origin in the plane of the parameters is now a connected region, and for such functions, Theorem III resumes a form more closely resembling the analogous theorem for analytic functions of two complex variables.

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* To admit negative values of y, let $f(x, y) = \dfrac{y}{x - \sqrt{|y|}}$.

SOME PROPERTIES OF POLYNOMIAL CURVES.

By Frank Irwin and H. N. Wright.

The first part of the present paper develops certain simple but important properties of the curve

$$y = F(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n, \tag{1}$$

a curve which, that we may have a name for it, we shall call a *polynomial curve.** The writers have been able to find but scant account of these curves in the literature. In any case they feel that the mere fact that such a book, for instance, as Loria's *Spezielle Kurven* gives no treatment of them is almost, in itself, a justification for a paper on the subject in this place.

The second part of the paper gives, partly in application of the preceding theory, graphic constructions for the imaginary roots of a cubic equation and of certain biquadratics, a treatment of the quadratic being included for the sake of completeness.

I. A line parallel to the y-axis at a distance $-a_1/na_0$ from it, that is, the line $x = -a_1/na_0$ has important properties with reference to the curve (1). $-a_1/na_0$ is the mean value of the roots of the equation $F(x) = 0$; or, in other words, the center of gravity of the n points, real and imaginary, in which the x-axis cuts the curve, lies on the line

$$x = -a_1/na_0.$$

Let us call this line the *axis* of the curve, the point where it cuts the curve its *vertex*. Then if this axis be taken as the y-axis, the equation of the curve may be written:

$$y = f(x) \equiv p_0x^n + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_n, \tag{2}$$

with the term in x^{n-1} missing. We shall, unless the contrary is stated, suppose the equation of the curve to be taken in this form.†

Now the property just stated for the x-axis is true for any line.

Proposition I. *The center of gravity of the n points, real and imaginary, in which any straight line, not parallel to the axis of the curve, cuts a polynomial curve, lies on the axis, provided n is greater than* 2.

* We shall assume, once for all, that unless the contrary is stated, n is greater than 2.

† It appears at once that the curve $y = F^r(x)$, where $F^r(x)$ denotes the rth derivative, has the same axis.

For, if the equation of the line be $y = mx + b$, the abscissas of the intersections are the roots of

$$f(x) - mx - b = 0,$$

and the mean value of these roots is zero.

Thus the axis of the curve is the diameter corresponding to any system of parallel chords; or *it is the polar line, with reference to the curve, of each point on the line at infinity.**

This property of the axis is merely a special case of the much more general and decidedly striking property stated in the following proposition.

PROPOSITION II. *The center of gravity of the mn points of intersection, real and imaginary, of any algebraic curve, $G(x, y) = 0$, of order m, and the polynomial curve (2), lies, when it exists at all, that is when the points are all finite, on the axis of the latter curve, always provided n is greater than 2.*

The same will be true in certain other cases also of the center of gravity of the *finite* points of intersection.

The condition that none of the mn points of intersection should lie at infinity is evidently that the coefficient of y^m in $G(x, y)$ should not be zero. Supposing this to be the case, let us consider the equation, say $g(x) = 0$, for the abscissas of the intersections, obtained by substituting the value of y from (2) in $G(x, y) = 0$. The result of this substitution in any term, cx^hy^k, of the latter equation will be

$$cx^h(p_0{}^kx^{nk} + kp_0{}^{k-1}p_2x^{n(k-1)} \cdot x^{n-2} + \text{terms of lower degree in } x);$$

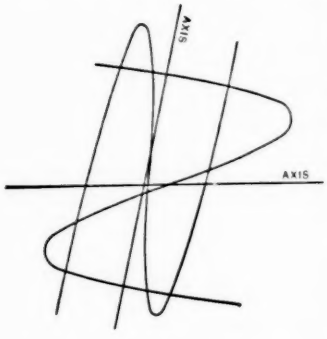
that is, the two highest terms in x will be of degrees $h + nk$ and $h + nk - 2$. In particular the term of $G(x, y)$ containing y^m, say cy^m $(c \neq 0)$, will give rise to $cp_0{}^mx^{mn} + cmp_0{}^{m-1}p_2x^{mn-2} + \cdots$; while the highest possible power of x derived from any other term of $G(x, y)$ will be the $(mn - n + 1)$th—coming from a term in xy^{m-1} (if present)—a power less than the $(mn - 1)$th, on our hypothesis that n is greater than 2. The equation $g(x) = 0$, then, will begin with $cp_0{}^mx^{mn}$, while the term in x^{mn-1} will certainly be lacking; so that the sum of the roots of this equation will be zero; which proves the proposition for this, the general case.

If, however, the term in y^m be absent from $G(x, y)$, the case will be more complicated. The degree of $g(x)$ will not now be mn; geometrically, some of the points of intersection will have gone off to infinity. The

* This suggests the question—what is the most general class of algebraic curves having such a property, the property, namely, that there exists a line l_1 each and every point on which has one and the same polar line l_2 with respect to the curve? If we project the line l_1 into the line at infinity, the line l_2 into the y-axis, it may readily be seen that the curve will reduce to the form $p_0x^n +$ (terms of degree less than $n - 1$ in x and y) $= 0$; so that the class of curves in question are the projections of curves having equations of this type.

proposition may still hold with reference to the finite points of intersection. An example will indicate sufficiently how we should proceed.

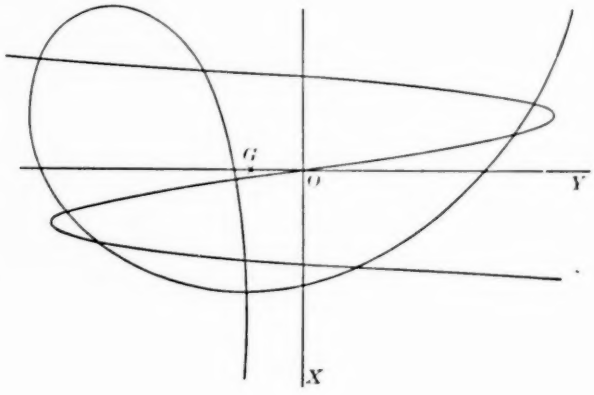
Suppose $G(x, y) \equiv ax^{10}y^2 + bx^6y^3 + cx^2y^4 + dxy^5 + e$, and let $n = 3$. *Two* terms, the first and fourth, will furnish the maximum value of $h + nk$, viz., 16. $g(x)$, then, will begin with $(ap_0^2 + dp_0^5)x^{16}$. Unless a and d are such that the coefficient of x^{16} vanishes, our proposition will not hold, for the second term of $g(x)$ will be a term in x^{15} derived from bx^6y^3; if, however, this term be not present (if $b = 0$), then the proposition holds.



A particular case in which the proposition holds is where the curve $G(x, y) = 0$ is itself a polynomial curve, if only the degree of the polynomial be less than $n - 1$. The case where it is a straight line (our first proposition) comes under this head.

Further, the center of gravity in the case of two polynomial curves whose axes are not parallel lies at the intersection of their axes (see the figure). For all their intersections are finite.

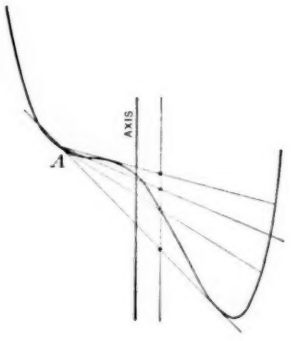
The next figure illustrates the intersections of a cubic polynomial curve, $80y = 9(3x^3 - 64x)$, with an oblique strophoid (also a cubic). It



is not a perfectly simple thing to get two such curves all of whose intersections are real. The reader may, if he wish to, verify on the drawing the fact that the sum of the abscissas of the 5 points of intersection with positive abscissas is equal to that of the 4 points with negative abscissas, and that so the center of gravity, G, actually does lie on the axis of the polynomial curve (the y-axis in this case).

Another curious property is the following:

PROPOSITION III. *If through a given point on a polynomial curve a variable straight line be drawn, the locus of the center of gravity of the $n-1$ other points in which it cuts the curve is a straight line parallel to the axis of the curve* ($n > 2$).



This is evident since, calling the given point on the curve A, the center of gravity in question G, and the center of gravity of all the n points in which the straight line cuts the curve G', we have $AG/AG' = n/(n-1)$; while the locus of G' is the axis of the curve.

From this or from Proposition I follows the construction for a tangent of a cubic polynomial curve: *to draw the tangent at a point* (x_0, y_0) *of such a curve, the axis of the curve being supposed to be taken as the y-axis, join the point to that point of the curve whose abscissa is* $-2x_0$.

It also appears that *a cubic polynomial curve is symmetric with respect to its point of inflection.*

PROPOSITION IV. *A polynomial curve of order n is also of class n, and (a) the sum of the abscissas of the points of contact of the n tangents, real and imaginary, that may be drawn to it from any point, as well as (b) the sum of their slopes, is independent of the ordinate of the given point.**

Let (h, k) be the point from which the tangents are drawn, and (x_i, y_i) any one of the points of contact. The condition that the tangent at (x_i, y_i) should go through (h, k) is $k - y_i = m_i(h - x_i)$, m_i denoting the slope of the curve at the point of contact. Substituting herein the values of y_i and m_i in terms of x_i, we see that x_i is a root of

$$(n-1)p_0x^n - nhp_0x^{n-1} + (n-3)p_2x^{n-2} + \cdots - 2hp_{n-2}x + k - hp_{n-1} - p_n = 0. \qquad (3).$$

The sum of the abscissas of the points of tangency is the sum of the roots of (3). But this sum is seen to be independent of k, which proves (a) of the proposition. In fact, since the constant term is the only coefficient in (3) which depends upon k, we see that all of the elementary symmetric functions of the abscissas of the points of tangency, except their product, are independent of the ordinate of the point from which the tangents are drawn. To prove (b) of the proposition, since

$$m_i = np_0x_i^{n-1} + (n-2)p_2x_i^{n-3} + (n-3)p_3x_i^{n-4} + \cdots + 2p_{n-2}x_i + p_{n-1},$$

we have

* This proposition holds also when $n = 2$.

$$\Sigma m_i = np_0s_{n-1} + (n-2)p_2s_{n-3} + (n-3)p_3s_{n-4} + \cdots + 2p_{n-2}s_1 + np_{n-1},$$

where s_j stands for Σx_i^j, and the summation is extended to include all the points of contact. From Newton's formula,

$$q_0s_j + q_1s_{j-1} + \cdots + q_{j-1}s_1 + jq_j = 0,$$

$q_0, q_1, \cdots, q_n$ standing for the coefficients of (3), it is seen that s_j may be expressed rationally in terms of $q_0, q_1, \cdots, q_j$. Therefore Σm_i, which involves $s_1, s_2, \cdots, s_{n-1}$, but not s_n, may be expressed in terms of $q_0, q_1, \cdots, q_{n-1}$, all of which, as already noticed, are independent of k.*

For the particular case of the *quartic* polynomial curve, and for a point on the axis ($h = 0$), Prop. IV holds for the *real* tangents that may be drawn to the curve considered by themselves:† in fact, *tangents at points equidistant from the axis intersect on the axis, and the sum of their slopes is constant and equal to* $2p_3$. For, in the first place, equation (3) reduces in this case to

$$3p_0x^4 + p_2x^2 + k - p_4 = 0,$$

an equation which, if it is satisfied by x_i, is also satisfied by $-x_i$; and, secondly, the sum of the slopes of the tangents at points of the curve whose abscissas are x_i and $-x_i$ is, since

$$m_i = 4p_0x_i^3 + 2p_2x_i + p_3,$$

equal to $2p_3$. If there are points on the axis from which four real tangents can be drawn, the sum of the slopes is different for them, namely $4p_3$.

In particular, the abscissas of the two inflections, real or imaginary, being, by the second footnote to this paper, equal and of opposite sign, *the inflectional tangents meet on the axis.*

Again, the points of contact of the real double tangent, if the curve have one, are equidistant from the axis and its slope is p_3. Since this is the slope at the vertex, we have here a means of constructing, from its graph, the axis of a quartic of this kind: we find the point where the tangent is parallel to the bitangent; this is the vertex, a point on the axis.

We may include here for convenience one further property of the quartic. *The points of intersection with the curve of any line of slope* p_3, $y = p_3x + b$, *consist of pairs of points equidistant from the axis.* For the abscissas of these intersections are the roots of

* We may state a proposition still more definite than the above, which if it adds nothing very important is at all events picturesque: *as* (h, k) *moves on a line parallel to the y-axis, the center of gravity of the points of contact of the tangents from* (h, k) *moves on a parallel line with a velocity* $1/(n-1)$ *times as great.* For its ordinate, $\bar{y} = \Sigma y_i/n = (p_0s_n + p_2s_{n-2} + \cdots + np_n)/n$. Here s_n alone involves k, and $q_0s_n = -q_1s_{n-1} - \cdots - nq_n$. So that $\bar{y}$ = constant $+ p_0s_n/n$ = constant $- q_np_0/q_0$ = constant $- k/(n-1)$.

† That this is not true in general may be seen easily by examining a few simple cases.

$$p_0x^4 + p_2x^2 + p_4 - b = 0,$$

which consist of pairs of numbers equal and of opposite sign. We might call the direction of these lines of slope p_3 *the direction of symmetry* of the curve.

Note. It is of interest to state the projective propositions of which ours are special metrical cases.* The polynomial curve projects into the most general curve, C, with a singular point, A, of order $n - 1$ and all the branches there tangent to one and the same straight line. On the other hand, as mentioned towards the beginning of this article, the polar line of the point at infinity on a given line cuts that line in the center of gravity of the latter's intersections with the curve. Our propositions may then be generalized as follows:

PROPOSITION I. *All the points on the tangent at A have one and the same polar line with respect to C, a line that passes through A.*

PROPOSITION II. *Join to A the intersections of C with any algebraic curve, G, of order m (that does not pass through A). Then all the points on the tangent at A have one and the same polar line with respect to this pencil of mn lines, and this polar line is independent of what curve G is chosen.*

The generalization of III offers no difficulty.

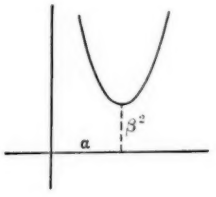
PROPOSITION IV (a). *If the points of contact of the n tangents from a point P to the curve C be joined to A, all points on the tangent at A have the same polar line with respect to this pencil of n lines, and the polar line does not vary so long as P stays on a given line through A.*

II. *Construction, from the graph, of the complex roots of a quadratic, cubic, biquadratic equation.*†

Quadratic. Let $\alpha + \beta i$ be the complex roots of a quadratic equation,

$$f(x) \equiv x^2 - 2\alpha x + \alpha^2 + \beta^2 = 0,$$

and suppose the graph of $y = f(x)$ constructed. Then $dy/dx = 2x - 2\alpha$, which vanishes when $x = \alpha$, while $y = \beta^2$ when $x = \alpha$. Therefore *the abscissa of the minimum point is the real part of the rootsand the ordinate is the square of the coefficient of* $\pm i$.



Cubic. Consider a cubic equation with two complex roots $\alpha \pm \beta i$,

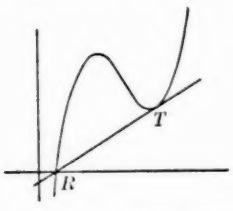
$$f(x) \equiv (x - \rho)(x^2 - 2\alpha x + \alpha^2 + \beta^2) = 0.$$

Let the graph of $y = f(x)$ cut the x-axis in R, then, by Prop. III, the center

* Something of the kind was suggested to us by Professor Veblen.

† See Popular Astronomy for February, 1909, where Mr. R. H. Gleason treats the quadratic and gives the construction in the text for the cubic.

of gravity of the second and third intersections of any line through R with the curve lies in a line parallel to the y-axis. Now, first, the equation of this line is $x = \alpha$, for the x-axis is a line through R cutting the cubic also where $x = \alpha \pm i\beta$, whose mean is α. And, secondly this line must pass through T, the point of contact of the tangent from R: in other words the abscissa of T is α.



Again, the slope of the curve, $x^2 - 2\alpha x + \alpha^2 + \beta^2 + 2(x - \alpha)(x - \rho)$, is β^2 when $x = \alpha$.

Hence, *the abscissa of T is the real part of the complex roots and the slope at T is the square of the coefficient of $\pm i$.*

Biquadratic. Our discussion will be confined to biquadratics with two real and two complex roots, say, $\alpha \pm \beta$ and $\gamma \pm \delta i$:

$$f(x) \equiv x^4 - 2(\alpha + \gamma)x^3 + (\alpha^2 - \beta^2 + \gamma^2 + \delta^2 + 4\alpha\gamma)x^2$$
$$- 2(\alpha\gamma^2 + \alpha\delta^2 + \alpha^2\gamma - \beta^2\gamma)x + (\alpha^2 - \beta^2)(\gamma^2 + \delta^2) = 0.$$

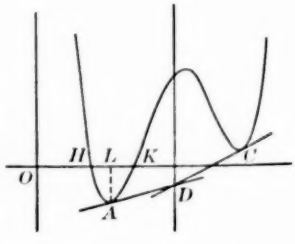
The axis of $y = f(x)$ is $x = (\alpha + \gamma)/2$. Then the points where $x = \alpha$, γ will, referred to the axis of the curve as y-axis, have as abscissas $(\alpha - \gamma)/2$ and $(\gamma - \alpha)/2$ respectively. Therefore, since these abscissas are equal and of opposite sign, the tangents at these points will, as proved above, meet on the axis of the curve. We find, furthermore, that when $x = \alpha$, $y = -\beta^2(\alpha - \gamma)^2 - \beta^2\delta^2$, $dy/dx = -2\beta^2(\alpha - \gamma)$. The equation of the tangent at this point of the curve is, therefore,

$$y + \beta^2(\alpha - \gamma)^2 + \beta^2\delta^2 = -2\beta^2(\alpha - \gamma)(x - \alpha);$$

and where this intersects the axis of the curve, $y = -\beta^2\delta^2$.

In constructing the roots of the equation, the axis of the curve is supposed to be known. It has been shown above how this may be constructed from the graph if the curve has a bitangent; otherwise we may construct it from the equation of the curve.

Let the curve cut the x-axis in H and K. Bisect HK at L. Then $OL = \alpha$, $LK = \beta$. Draw the tangent at A, the point of the curve whose abscissa is α, and let it intersect the axis of the curve at D. From D draw DC tangent to the curve (A and C are equidistant from the axis: this distinguishes DC from other possible tangents from D). Then the abscissa of C is γ, and the ordinate of D is $-\beta^2\delta^2$, from which δ may be found.



COMPARISON THEOREMS FOR HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF GENERAL ORDER.*

By R. D. Carmichael.

Introduction.

The fundamental notion of adjoint systems of boundary conditions associated with adjoint linear differential equations of order n, first defined in a general manner by Birkhoff,† is fruitful of interesting consequences concerning the relation between solutions of two linear differential equations. In particular it leads to certain extensions of one of Sturm's classical theorems of comparison‡ to equations of order higher than the second. The purpose of this paper is to set forth some of the results which thus emerge. The theorems obtained may be compared with certain other related results which are likewise of the nature of Sturmian theorems.§

After deriving preliminary formulæ concerning Lagrange adjoint differential equations in § 1 and concerning Birkhoff adjoint linear forms in § 2 (for which no essential novelty is claimed), I give in § 3 the two general theorems of comparison of the paper. These are of considerable generality and may be employed in an easy way to derive numerous more special theorems of comparison. In particular, it is pointed out that the first of these general theorems contains as a special case Sturm's classical theorem of comparison for linear homogeneous binomial equations of the second order. In § 4 I give four of the more elegant specific consequences of the general theorems of comparison. Finally, in § 5 I consider briefly a case in which one of the two equations involved has special form and show by an example how the theorem obtained may be employed to yield information concerning the zeros of solutions of certain

* Presented to the American Mathematical Society (Chicago), April, 1917.

† G. D. Birkhoff, Transactions of the American Mathematical Society, vol. 9 (1908), pp. 373–395. See also, in the same journal, a further treatment of adjoint systems by M. Bôcher, vol. 14 (1913), pp. 403–420, and by D. Jackson, vol. 17 (1916), pp. 418–424. As a matter of fact it is possible to state our principal results without reference to Birkhoff's notion of adjoint conditions; but that notion was the guide in the investigation and illuminates the results obtained.

‡ C. Sturm, Journal de mathématiques, vol. 1 (1836), pp. 106–186. See also M. Bôcher, Bulletin of the American Mathematical Society, ser. 2, vol. 4 (1898), pp. 295–313, 365–376.

§ See G. D. Birkhoff, *Annals of Mathematics*, vol. 12 (1911), pp. 103–127; A. Davidoglou, Annales de l'École Normale Supérieure, ser. 3, vol. 22 (1905), pp. 539–565; O. Haupt, Dissertation, Würzburg, 1911.

types of equations. The general treatment of the matter raised by this example is reserved for later consideration.

§ 1. Preliminary formulæ.

In connection with the linear differential expression

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u \tag{1}$$

let us consider also the Lagrange adjoint expression

$$M(v) \equiv (-1)^n \frac{d^n}{dx^n}(l_n v) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(l_{n-1} v) + \cdots + \frac{d}{dx}(l_1 v) + l_0 v.$$

Arranging the latter according to the derivatives of v we write it in the form

$$M(v) \equiv m_n \frac{d^n v}{dx^n} + m_{n-1} \frac{d^{n-1} v}{dx^{n-1}} + \cdots + m_1 \frac{dv}{dx} + m_0 v, \tag{2}$$

where

$$\begin{aligned}
(-1)^n m_n &= l_n, \\
(-1)^{n-1} m_{n-1} &= l_{n-1} - n l_n', \\
(-1)^{n-2} m_{n-2} &= l_{n-2} - (n-1) l_{n-1}' + \frac{(n-1)n}{2!} l_n'', \\
(-1)^{n-3} m_{n-3} &= l_{n-3} - (n-2) l_{n-2}' \\
&\quad + \frac{(n-2)(n-1)}{2!} l_{n-1}'' - \frac{(n-2)(n-1)n}{3!} l_n''', \\
&\cdots\cdots\cdots

If we write

$$\begin{aligned} R(u, v) \equiv [l_n v]\frac{d^{n-1}u}{dx^{n-1}} &+ \left[l_{n-1}v - \frac{d}{dx}(l_n v) \right]\frac{d^{n-2}u}{dx^{n-2}} \\ &+ \left[l_{n-2}v - \frac{d}{dx}(l_{n-1}v) + \frac{d^2}{dx^2}(l_n v) \right]\frac{d^{n-3}u}{dx^{n-3}} + \cdots \\ &+ \left[l_1 v - \frac{d}{dx}(l_2 v) + \frac{d^2}{dx^2}(l_3 v) - \cdots \right. \\ &\left. + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(l_n v) \right] u, \end{aligned} \tag{4}$$

then it is well known* and easily verified that

$$vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u, v)\}. \tag{5}$$

We now introduce also the forms

$$P(y) \equiv p_n\frac{d^n y}{dx^n} + p_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1\frac{dy}{dx} + p_0 y \tag{6}$$

and

$$D(y) \equiv P(y) - M(y). \tag{7}$$

Here we assume that the functions p_n, p_{n-1}/p_n, p_{n-2}/p_n, $\cdots$, p_0/p_n of x are continuous in (ab).

We shall uniformly employ the notation u_k, v_k, y_k, for various subscripts k, to denote particular solutions of the differential equations

$$L(u) = 0, \qquad M(v) = 0, \qquad P(y) = 0, \tag{8}$$

respectively. From the relations $L(u_s) = 0$ and $P(y_t) = 0$ we have $y_t L(u_s) - u_s P(y_t) = 0$; whence, in view of (5) and (7), it follows readily that

$$\frac{d}{dx}\{R(u_s, y_t)\} - D(y_t)u_s = 0. \tag{9}$$

Thence, if α and β are in (ab), we have

$$R(u_s, y_t)\Big|_{x=\alpha}^{x=\beta} = \int_\alpha^\beta D(y_t)u_s dx. \tag{10}$$

§ 2. Adjoint linear forms associated with adjoint linear differential equations.

Let us denote the kth derivative with respect to x of any function g by $g^{(k)}$ with the understanding that $g^{(0)} \equiv g$. Let α and β be any two

* See, for instance, Forsyth's Theory of Differential Equations, vol. IV, p. 252.

points in (ab) and take $\alpha < \beta$. By $W_1, W_2, \cdots, W_{2n}$ we denote any $2n$ linearly independent homogeneous forms

$$W_i \equiv W_i(u) = \sum_{j=1}^{n} c_{ij} u^{(j-1)}(\alpha) + \sum_{j=n+1}^{2n} c_{ij} u^{(j-n-1)}(\beta), \quad i = 1, 2, \cdots, 2n, \quad (11)$$

in the $2n$ arguments $u(\alpha), u'(\alpha), \cdots, u^{(n-1)}(\alpha), u(\beta), u'(\beta), \cdots, u^{(n-1)}(\beta)$. We denote by $\overline{D}$ the determinant of these forms:

$$\overline{D} = |c_{ij}|.$$

With the linear forms (11) Birkhoff (l. c., p. 375) has associated a second set of linear homogeneous forms $V_1, V_2, \cdots, V_{2n}$ in the same $2n$ arguments which we may call the *set adjoint to* $W_1, W_2, \cdots, W_{2n}$ *with respect to the differential expression* $L(u)$ *and the points* α *and* β. They are defined by the relation

$$\sum_{i=1}^{2n} W_i(u) V_i(y) \equiv R(u, y) \Big|_{x=\alpha}^{x=\beta} \quad (12)$$

which is to subsist as an identity in u and y and their derivatives at the points α and β.

It is clear that the second member of (12) is a bilinear form in the two sets of variables $u(\alpha), u'(\alpha), \cdots, u^{(n-1)}(\alpha), u(\beta), u'(\beta), \cdots, u^{(n-1)}(\beta)$ and $y(\alpha), y'(\alpha), \cdots, y^{(n-1)}(\alpha), y(\beta), y'(\beta), \cdots, y^{(n-1)}(\beta)$. In view of (4) we may write

$$R(u, y) \Big|_{x=\alpha}^{x=\beta} = \sum_{i,j=1}^{n} r_{ij} u^{(i-1)}(\alpha) y^{(j-1)}(\alpha) + \sum_{i,j=n+1}^{2n} r_{ij} u^{(i-n-1)}(\beta) y^{(j-n-1)}(\beta), \quad (13)$$

where the coefficients r_{ij} are independent of u and y but in the first summation are functions of α and in the second are functions of β. The quantities r_{ij}, $i, j = 1, 2, \cdots, 2n$, not entering into (13), we take to be zero. Then we see that $r_{ij} = 0$ when either subscript is greater than n and the other is not greater than n. Also, from (4) it follows that $r_{ij} = 0$ if $i \leqq n$, $j \leqq n$ and $i + j > n + 1$; and again that $r_{ij} = 0$ if $i > n$, $j > n$ and $i + j - 2n > n + 1$. By R we denote the determinant of the bilinear form (13); we have

$$R = |r_{ij}| = \{l_n(\alpha) l_n(\beta)\}^n, \quad (14)$$

a value which is different from zero since l_n does not vanish in $(\alpha\beta)$.

That the forms V are indeed uniquely determined by (12) follows from the fact that $\overline{D} \neq 0$. We denote them by

$$V_i(y) = \sum_{j=1}^{n} \gamma_{ij} y^{(j-1)}(\alpha) + \sum_{j=n+1}^{2n} \gamma_{ij} y^{(j-n-1)}(\beta), \qquad i = 1, 2, \cdots, 2n, \quad (15)$$

and their determinant by Δ:

$$\Delta = |\gamma_{ij}|.$$

It is easy to see that we have

$$\sum_{i=1}^{2n} c_{ij}\gamma_{is} = r_{js}, \qquad j, s = 1, 2, \cdots, 2n. \tag{16}$$

Thence it follows that $\bar{D}\Delta = R$. Hence $\Delta \neq 0$ and the forms V are therefore linearly independent. Clearly, then, the forms $W_1, \cdots, W_{2n}$ are adjoint to the forms $V_1, \cdots, V_{2n}$, so that the relation between the W's and the V's is reciprocal. Moreover, the coefficients γ_{is} may be determined from (16) in terms of the coefficients c_{ij} and *vice versa*.

In accordance with Birkhoff's definition (l. c., p. 375), as extended by Bôcher (l. c., p. 418), we shall say that the two sets of boundary conditions

$$W_\sigma(u) = 0, \qquad V_\rho(y) = 0, \tag{17}$$

where σ runs over m numbers of the set $1, 2, \cdots, 2n$ and ρ runs over the remaining $2n - m$ numbers of this set, are *adjoint with respect to the differential expression $L(u)$ and the points α and β*. The most important case, namely that in which $m = n$, is the one of Birkhoff's original definition.

In the case when $m = n$, if an interchange of y and u in (17) merely interchanges the two sets of conditions* we say that the boundary conditions $W = 0$ in (17) are *self-adjoint*. If an interchange of y and u and of α and β in (17) interchanges the two sets of conditions* we say that the boundary conditions $W = 0$ in (17) are *skew-self-adjoint*.†

It is usually desirable to choose the notation so that σ in (17) runs over the first m or the last m integers of the set $1, 2, \cdots, 2n$ and ρ over the remaining $2n - m$ of these integers. If we assume for the moment that σ runs over the set $1, 2, \cdots, m$ we shall see from (16) that the second set of conditions in (17) depends only on the conditions $W_1 = 0, \cdots, W_m = 0$ and not at all on the remaining W's, in the sense that a variation in the choice of $W_{m+1}, \cdots, W_{2n}$ merely replaces the conditions $V_{m+1} = 0, \cdots, V_{2n} = 0$ by a set of equivalent conditions. Moreover, if the conditions $W_1 = 0, \cdots, W_m = 0$ are replaced by any set of equivalent conditions the resulting set of conditions $V_{m+1} = 0, \cdots, V_{2n} = 0$ are again equivalent to the old. For, in view of the conditions $W_1 = 0, \cdots, W_m = 0$ the suggested changes are all allowed for by replacing c_{ij} by c_{ij}', where

$$c_{ij}' = \sum_{k=1}^{m} d_{ik}c_{kj}, \qquad i = 1, 2, \cdots, m;$$

* It is not necessary that the individual equations shall be preserved; it is sufficient if the conditions are replaced by equivalent conditions.

† I believe that the notion of skew-self-adjoint conditions has not been employed before.

$$c_{ij}' = \sum_{k=m+1}^{2n} d_{ik}c_{kj}, \qquad i = m+1, \cdots, 2n.$$

In view of (16) we see that this is equivalent to replacing γ_{is} by γ_{is}' where

$$\gamma_{is}' = \sum_{k=1}^{m} d_{ki}\gamma_{ks}, \qquad i = 1, 2, \cdots, m;$$

$$\gamma_{is}' = \sum_{k=m+1}^{2n} d_{ki}\gamma_{ks}, \qquad i = m+1, \cdots, 2n.$$

We now define a particular pair of adjoint forms special cases of which we shall employ later: Let σ and τ be two integers (the same or different) of the set 0, 1, 2, $\cdots$, $n-1$. Then for the W's we take the following forms:

$$\left.\begin{aligned} W_i(u) &= u^{(i-1)}(\alpha) - k_i u^{(\sigma)}(\alpha), \qquad 1 \leqq i \leqq n, \qquad i \neq \sigma+1; \\ & \qquad\qquad W_{\sigma+1}(u) = u^{\sigma}(\alpha) - k u^{(\tau)}(\beta); \\ W_i(u) &= u^{(i-n-1)}(\beta) - k_i u^{(\tau)}(\beta), \qquad n+1 \leqq i \leqq 2n, \\ & \qquad i \neq n+\tau+1; \qquad W_{n+\tau+1}(u) = u^{(\tau)}(\beta). \end{aligned}\right\} \tag{18}$$

Here $k, k_1, k_2, \cdots, k_{2n}$ are constants. (Given any solution u of the equation $L(u) = 0$ and any points α and β in (ab), it is clear that σ and τ exist so that corresponding constants k may be found in such way that $W_i = 0$ for all values i of the set 1, 2, $\cdots$, $2n$ except $i = n+\tau+1$. It is this fact which suggests the use of this particular set of forms.) It is clear that these forms are linearly independent and of determinant unity. They are the special case of forms (11) in which

$$\left.\begin{aligned} c_{ii} &= 1; \qquad c_{i,\,\sigma+1} = -k_i, \qquad i \neq \sigma+1 \text{ and } i \leqq n; \\ c_{\sigma+1,\,n+\tau+1} &= -k; \qquad c_{i,\,n+\tau+1} = -k_i, \qquad i \neq n+\tau+1 \text{ and } i > n; \\ & \qquad c_{ij} = 0 \text{ in all other cases.} \end{aligned}\right\} \tag{19}$$

Let $V_1, V_2, \cdots, V_{2n}$ be the forms which are adjoint to the W's in (18). By means of (16) we see that these forms are the special case of (15) in which

$$\left.\begin{aligned} \gamma_{js} &= r_{js}, \quad j, s = 1, 2, \cdots, 2n, \quad j \neq \sigma+1, \quad j \neq n+\tau+1; \\ \gamma_{\sigma+1,\,s} &= \sum_{i=1}^{n} k_i r_{is}, \qquad s = 1, 2, \cdots, 2n, \qquad k_{\sigma+1} = 1; \\ \gamma_{n+\tau+1,\,s} &= \sum_{i=n+1}^{2n} k_i r_{is} + k\sum_{i=1}^{n} k_i r_{is}, \qquad s = 1, 2, \cdots, 2n, \qquad k_{n+\tau+1} = 1. \end{aligned}\right\} \tag{20}$$

These values reduce to simpler forms for certain subscripts s in view of the fact that many of the quantities r_{ij} have the value zero, as we saw above.

Let us consider the special case in which $\sigma = n - 1$, $\tau = 0$, $k_i = 0$ ($i \neq n$ or $n + 1$), $k = 0$. Then the conditions $W_\sigma = 0$, $\sigma = 1, 2, \cdots, n - 1, n + 1$, are

$$u(\alpha) = 0,\ u'(\alpha) = 0,\ \cdots,\ u^{(n-2)}(\alpha) = 0,\ u(\beta) = 0. \tag{21}$$

For this special case it is easy to see through either (20) or (12) that the adjoint conditions are equivalent to

$$y(\alpha) = 0,\ y(\beta) = 0,\ y'(\beta) = 0,\ \cdots,\ y^{(n-2)}(\beta) = 0, \tag{22}$$

the first of these being precisely $V_n = 0$ and the others on account of the relation $V_n = 0$ being equivalent to the remaining set of conditions $V_{n+2} = 0, \cdots, V_{2n} = 0$. In (21) and (22) we have a simple example of skew-self-adjoint boundary conditions. These conditions, with the exception of $y(\alpha) = 0$, are employed in the first theorem of § 4 and similar sets in the remaining theorems of that section.

When the reader has reached the first and second theorems of § 4 he will see that several analogous theorems may be obtained through the use of certain skew-self-adjoint conditions based on special cases of the forms (18) similar to those introduced in the last preceding paragraph.

§ 3. General theorems of comparison.

We are now in position readily to prove the following theorem:

FIRST GENERAL THEOREM OF COMPARISON. *Let u_1 be any solution of the equation $L(u) = 0$ which does not vanish in the interior of an interval $(\alpha\beta)$ lying in (ab); for definiteness say that it is positive* in the interior of $(\alpha\beta)$. Let y_1 be a solution of the equation $P(y) = 0$ so related to u_1 that*

$$R(u_1, y_1)\Big|_{x=a}^{x=\beta} \leqq 0. \tag{23}$$

Suppose that the coefficients of $L(u)$ and $P(y)$ are so related that $p_k - m_k$ does not change sign in $(\alpha\beta)$ for any k of the set $0, 1, 2, \cdots, n$. Then, if p_k is different from m_k for some k of the set $0, 1, 2, \cdots, n$ and some x of $(\alpha\beta)$, at least one of the quantities $y_1^{(k)}$, $k = 0, 1, 2, \cdots, n$, for which the corresponding $p_k - m_k$ is different from zero at some point in $(\alpha\beta)$, vanishes in the interior of $(\alpha\beta)$, provided that, for every k for which $p_k - m_k \not\equiv 0$ in $(\alpha\beta)$, the two functions

$$p_k - m_k \text{ and } y_1^{(k)}$$

have the same sign at a point P_k in $(\alpha\beta)$.

* It is obvious that we have a like result obtained by giving to u and y signs opposite to those used in the theorem.

The point P_k may well be taken at α or at β if $p_k - m_k$, when not identically zero in $(\alpha\beta)$, is different from zero at either of these points.

It is obvious that condition (23) may be replaced by

$$\sum_{i=1}^{2n} W_i(u_1)V_i(y_1) \leqq 0, \tag{24}$$

where $W_1, \cdots, W_{2n}$ and $V_1, \cdots, V_{2n}$ are any two sets of adjoint forms, in the sense of the definition in § 2. This condition is satisfied in particular if W_i and V_i have opposite signs when neither of them is zero. As a very special case, it is satisfied if u_1 and y_1 satisfy adjoint boundary conditions defined for the interval $(\alpha\beta)$ and with reference to the differential expression $L(u)$.

The proof of the foregoing theorem is based on equation (10), which for the case before us may be written in the form

$$\begin{aligned} R(u_1, y_1)\Big|_{x=\alpha}^{x=\beta} = &\int_\alpha^\beta (p_n - m_n)y_1^{(n)}u_1 dx \\ &+ \int_\alpha^\beta (p_{n-1} - m_{n-1})y_1^{(n-1)}u_1 dx + \cdots \\ &+ \int_\alpha^\beta (p_1 - m_1)y_1' u_1 dx + \int_\alpha^\beta (p_0 - m_0)y_1 u_1 dx. \end{aligned} \tag{25}$$

We now obtain the proof by showing that (25) yields a contradiction if we assume that no one of the quantities $y_1^{(k)}$, $k = 0, 1, 2, \cdots, n$, for which the corresponding $p_k - m_k$ is different from zero at some point in $(\alpha\beta)$, vanishes in the interior of $(\alpha\beta)$. Now by the hypothesis of the theorem each integrand in (25) whose factor $p_k - m_k$ is not identically zero in $(\alpha\beta)$ is positive at some point in $(\alpha\beta)$. From the hypothesis and the assumption just made it follows that no one of these integrands can change sign in $(\alpha\beta)$. Hence at least one of the integrals in (25) has a positive value and no one of them has a negative value. But this contradicts (23). On account of this contradiction we conclude to the truth of the theorem.

In a similar manner we prove the following:

SECOND GENERAL THEOREM OF COMPARISON. *Let y_2 be any solution of the equation $P(y) = 0$ for which $y_2^{(k)}$ and $p_k - m_k$ $(k = 0, 1, 2, \cdots, n)$ have the same sign in an interval $(\alpha\beta)$ of (ab) at every point for which neither is zero. Let u_2 be a solution of the equation $L(u) = 0$ which is positive at some point in $(\alpha\beta)$ and for which we have*

$$R(u_2, y_2)\Big|_{x=\alpha}^{x=\beta} \leqq 0 \qquad \left(\text{or } \sum_{i=1}^{2n} W_i(u_2)V_i(y_2) \leqq 0\right). \tag{26}$$

Then if p_k is not identically equal to m_k for every k of the set $0, 1, 2, \cdots, n$, u_2 vanishes in the interior of $(\alpha\beta)$.

This result follows at once from the relation obtained from (25) by replacing y_1 and u_1 throughout by y_2 and u_2, the method being to assume that u_2 does not vanish in the interior of $(\alpha\beta)$ and thus arrive at a contradiction of (26).

We point out some special cases of the foregoing theorems.

Note that the points α and β in the first of the theorems may be in particular two consecutive zeros of u_1.

In case $p_k \equiv m_k$ in $(\alpha\beta)$ for every k of the set 0, 1, 2, $\cdots$, n, except $k = s$, it follows from the first of the foregoing general theorems that $y_1^{(s)}$ has a root in the interior of $(\alpha\beta)$. In particular, if $p_k \equiv m_k$ for $k = 1, 2, \cdots, n$, but $p_0 \not\equiv m_0$ in $(\alpha\beta)$ then y_1 has a root between α and β.

The last particular case of our general theorem contains as a very special instance the following classical theorem of Sturm:

Let u_1 be a solution of the equation $u'' + l_0 u = 0$. Denote by α and β two consecutive zeros of u_1. Let y_1 be a solution of the equation $y'' + p_0 y = 0$. Then y_1 vanishes between α and β provided that $p_0 \geqq l_0$ in $(\alpha\beta)$, the equality sign not holding throughout $(\alpha\beta)$.

If y_1 does not vanish between α and β it is of one sign between α and β. We shall therefore suppose that u_1 and y_1 are both positive between α and β and show that (23) then holds, a result which requires that y_1 vanish between α and β in accordance with the foregoing general theorem of comparison. (Observe that m_0 is now equal to l_0.) We have

$$R(u_1, y_1) = u_1' y_1 - u_1 y_1'.$$

Hence (23) reduces to $u_1'(\beta) y_1(\beta) - u_1'(\alpha) y_1(\alpha) \leqq 0$. That this inequality is now verified is easily shown by observing that $u_1'(\alpha)$ is positive and $u_1'(\beta)$ is negative while $y_1(\alpha)$ and $y_1(\beta)$ are positive or zero.

In the second of the foregoing theorems we may take α and β as two consecutive zeros of y_2 provided that $p_k \equiv m_k$ in $(\alpha\beta)$ for $k = 1, 2, \cdots, n$ but $p_0 \not\equiv m_0$ in $(\alpha\beta)$. The theorem asserts then that u_2 vanishes between the two consecutive zeros α and β of y_2. It is easy to formulate in a similar way less special cases of the same theorem.

§ 4. Certain specific theorems of comparison.

By means of the first general theorem of comparison we may readily prove the following:

FIRST SPECIFIC THEOREM OF COMPARISON. *Let α be a point in (ab) and let u_3 be a solution of the equation $L(u) = 0$ for which $u_3(\alpha) = 0$, $u_3'(\alpha) = 0$, $\cdots$, $u_3^{(n-2)}(\alpha) = 0$. Let β be the leftmost point (assumed existent) to the right of α in (ab) for which $u_3(\beta) = 0$. Let y_3 be a solution of the equation $P(y) = 0$ for which $y_3(\beta) = 0, y_3'(\beta) = 0, \cdots, y_3^{(n-2)}(\beta) = 0$.*

Then, if, for $k = 0, 1, 2, \cdots, n$, *we have*

$$p_k \geqq m_k \text{ or } p_k \leqq m_k \text{ in } (\alpha\beta),$$

according as k *is even or odd, the equality sign not holding throughout* $(\alpha\beta)$ *for every* k, *at least one of the functions* $y_3^{(k)}$, $k = 0, 1, 2, \cdots, n$, *for which the corresponding difference* $p_k - m_k$ *is not identically zero in* $(\alpha\beta)$, *has a root in the interior of* $(\alpha\beta)$, *except when* $p_k \equiv m_k$ *for* $k = 0, 1, 2, \cdots, n-1$, *in which case either* $y_3^{(n)}$ *vanishes between* α *and* β *or all the functions* $y_3, y_3', y_3'', \cdots, y_3^{(n-1)}$ *vanish between* α *and* β.

Under the hypothesis it is clear that $u_3^{(n-1)}(\alpha)$ and $y_3^{(n-1)}(\beta)$ are different from zero. We assume (as we may without loss of generality) that u_3 is positive in the neighborhood of α to the right and that y_3 is positive in the neighborhood of β to the left. Then it is clear that u_3' is positive near α to the right, and thence in turn that u_3'', u_3''', $\cdots$, $u_3^{(n-1)}$ are positive near α to the right. In a similar way it may be seen that y_3' is negative near β to the left, y_3'' is positive near β to the left, y_3''' is negative, $\cdots$, $y_3^{(n-1)}$ is positive or negative near β to the left according as n is odd or even.

In view of (4) and the facts just adduced we see from the first general theorem of comparison that the above specific theorem is true if

$$l_n(\alpha)y_3(\alpha)u_3^{(n-1)}(\alpha) \geqq 0,$$

and hence if $y_3(\alpha) \geqq 0$. But if $y_3(\alpha) < 0$, y_3 has a root in the interior of $(\alpha\beta)$; whence it follows readily that the functions $y_3', y_3'', \cdots, y_3^{(n-1)}$ all have roots in the interior of $(\alpha\beta)$. Hence the theorem is true in any case.

It is easy to see that we have the following corollary:

COROLLARY. *If* $p_k - m_k \equiv 0$ *in* $(\alpha\beta)$ *for* $k = s+1, s+2, \cdots, n$, $s < n$, *then* $y_3^{(s)}$ *has a root in the interior of* $(\alpha\beta)$.

Again from the first general theorem of comparison we may derive the following:

SECOND SPECIFIC THEOREM OF COMPARISON. *Let* u_4 *be a solution of the equation* $L(u) = 0$ *for which* $u_4(\beta) = 0$, $u_4'(\beta) = 0$, $\cdots$, $u_4^{(n-2)}(\beta) = 0$. *Let* α *be the rightmost point (assumed existent) to the left of* β *in* (ab) *for which* $u_4(\alpha) = 0$. *Let* y_4 *be a solution of the equation* $P(y) = 0$ *for which* $y_4(\alpha) = 0$, $y_4'(\alpha) = 0, \cdots, y_4^{(n-2)}(\alpha) = 0$. *Then, if, for* $k = 0, 1, 2, \cdots, n$, *we have*

$$p_k \geqq m_k \text{ or } p_k \leqq m_k \text{ in } (\alpha\beta)$$

according as n *is even or odd, the equality sign not holding throughout* $(\alpha\beta)$ *for every* k, *at least one of the functions* $y_4^{(k)}$, $k = 0, 1, \cdots, n$, *for which the corresponding difference* $p_k - m_k$ *is not identically zero in* $(\alpha\beta)$, *has a root*

in the interior of $(\alpha\beta)$, *except when* $p_k \equiv m_k$ *for* $k = 0, 1, \cdots, n-1$, *in which case either* $y_4^{(n)}$ *vanishes between* α *and* β *or all the functions* $y_4, y_4', \cdots, y_4^{(n-1)}$ *vanish between* α *and* β.

Under the hypothesis it is clear that $u_4^{(n-1)}(\beta)$ and $y_4^{(n-1)}(\alpha)$ are different from zero. We assume (as we may without loss of generality) that u_4 is positive near β to the left and that y_4 is positive or negative near α to the right according as n is even or odd. Then $y_4', y_4'', \cdots, y_4^{(n-1)}$ are positive or negative near α to the right according as n is even or odd. Also, $u_4^{(k)}$, $k = 0, 1, \cdots, n-1$, is positive or negative near β to the left according as k is even or odd.

In view of these facts and equation (4) we see from the first general theorem of comparison that the foregoing specific theorem is true if

$$l_n(\beta) y_4(\beta) u_4^{(n-1)}(\beta) \leqq 0,$$

and hence if $y_4(\beta) \leqq 0$ when n is odd and if $y_4(\beta) \geqq 0$ when n is even. But if $y_4(\beta) > 0$ and n is odd or if $y_4(\beta) < 0$ and n is even, it is clear that y_4 has a zero in the interior of $(\alpha\beta)$; whence it follows readily that the functions $y_4', y_4'', \cdots, y_4^{(n-1)}$ vanish in the interior of $(\alpha\beta)$. Hence the theorem is true in any case.

COROLLARY. *If* $p_k - m_k \equiv 0$ *in* $(\alpha\beta)$ *for* $k = s+1, s+2, \cdots, n$, $s < n$, *then* $y_4^{(s)}$ *has a root in the interior of* $(\alpha\beta)$.

It is clear that other particular theorems analogous to the foregoing may readily be obtained in a similar manner through the use of skew-self-adjoint boundary conditions in which $n-1$ of the n conditions involve a single point. (Compare the last remark in § 3.)

From the second general theorem of comparison one readily derives the following result:

THIRD SPECIFIC THEOREM OF COMPARISON. *Let* α *be a point in* (ab) *and* y_5 *be a solution of the equation* $P(y) = 0$ *for which* $y_5(\alpha) = 0$, $y_5'(\alpha) = 0$, $\cdots$, $y_5^{(n-2)}(\alpha) = 0$. *Let* β *be the leftmost point (assumed existent) to the right of* α *in* (ab) *for which* Y *vanishes, where* Y *is the product of those functions* $y_5^{(k)}$ $(k = 0, 1, 2, \cdots, n)$ *for which the corresponding difference* $p_k - m_k$ *is not identically zero in* $(\alpha\beta)$, *the assumption being that such a product is existent. Furthermore, let every such difference* $p_k - m_k$ *(when not zero) be throughout* $(\alpha\beta)$ *of the same sign* as* $y_5^{(n-1)}(\alpha)$. *Let* u_5 *be a solution of the equation* $L(u) = 0$ *which is positive at some point in* $(\alpha\beta)$ *and satisfies the condition*

$$R(u_5, y_5)\Big|_{x=a}^{x=\beta} \leqq 0 \qquad \left[\text{or } \sum_{i=1}^{2n} W_i(u_5) V_i(y_5) \leqq 0\right].$$

Then this function u_5 *vanishes in the interior of* $(\alpha\beta)$.

* It should be observed that this sign is determined by the other conditions on y to within a simultaneous change of sign of all the quantities $y^{(k)}$, $k = 0, 1, 2, \cdots, n$.

For the proof it is sufficient to observe that the functions $y_5^{(k)}$ entering into the product Y have in the interior of $(\alpha\beta)$ the same sign as $y_5^{(n-1)}(\alpha)$.

In a similar manner the following result is obtained:

FOURTH SPECIFIC THEOREM OF COMPARISON. *Let β be a point in (ab) and let y_6 be a solution of the equation $P(y) = 0$ for which $y_6(\beta) = 0$, $y_6'(\beta) = 0$, $\cdots$, $y_6^{(n-2)}(\beta) = 0$. Let α be the rightmost point (assumed existent) to the left of β in (ab) for which Y_1 vanishes, where Y_1 is the product of those functions $y_6^{(k)}$ $(k = 0, 1, 2, \cdots, n)$ for which the corresponding difference $p_k - m_k$ is not identically zero in $(\alpha\beta)$, the assumption being that such a product is existent. Furthermore let every such difference $p_k - m_k$ (when not zero) be throughout $(\alpha\beta)$ of the same sign* as y_6^k in the neighborhood of β to the left. Let u_6 be a solution of the equation $L(u) = 0$ which is positive at some point in $(\alpha\beta)$ and satisfies the condition*

$$R(u_6, y_6)\Big|_{x=\alpha}^{x=\beta} \leqq 0 \qquad \left[\text{or } \sum_{i=1}^{2n} W_i(u_6)V_i(y_6) \leqq 0\right].$$

Then this function u_6 vanishes in the interior of $(\alpha\beta)$.

§ 5. Concerning a case in which P(y) has special form.

By means of the second general theorem of comparison we may readily prove the following:

THEOREM. *Let α and β be two consecutive zeros in (ab) of a solution y_7 of the equation $P(y) = 0$ and suppose that each of these zeros is of order $n - 1$. Suppose further that the products $y_7^{(k)}(p_k - m_k)$, $k = 0, 1, 2, \cdots, n$, are all non-negative or all non-positive in $(\alpha\beta)$. Then any solution u_7 of the equation $L(u) = 0$ vanishes between α and β provided that $p_k - m_k$ is not identically zero in $(\alpha\beta)$ for every k of the set $0, 1, 2, \cdots, n$ and provided that $p_0 - m_0$ is negative for some point in $(\alpha\beta)$ when n is even or*

$$l_n(\beta)y_7^{(n-1)}(\beta)u_7(\beta) \leqq l_n(\alpha)y_7^{(n-1)}(\alpha)u_7(\alpha), \tag{27}$$

when n is odd.

We shall prove this by supposing that u_7 does not vanish between α and β and thus obtaining a contradiction. We then take u_7 to be positive between α and β. We choose the sign of y_7 so that the products $y_7^{(k)}(p_k - m_k)$ are all non-negative in $(\alpha\beta)$. We have

$$\begin{aligned} R(u_7, y_7)\Big|_{x=\alpha}^{x=\beta} &= (-1)^{n-1}l_n y_7^{(n-1)}u_7\Big|_{x=\alpha}^{x=\beta} \\ &= (-1)^{n-1}[l_n(\beta)y_7^{(n-1)}(\beta)u_7(\beta) - l_n(\alpha)y_7^{(n-1)}(\alpha)u_7(\alpha)]. \end{aligned} \tag{28}$$

* It should be observed that this sign is determined by the other conditions on y to within a simultaneous change of sign of all the quantities $y^{(k)}$, $k = 0, 1, 2, \cdots, n$.

From the second general theorem of comparison it follows that for the conclusion of the theorem it is sufficient that the last member of (28) shall be not greater than zero. When n is odd this condition is necessitated by (27). That the condition is satisfied when n is even follows from the fact that we then have $y_7^{(n-1)}(\alpha)$ negative and $y_7^{(n-1)}(\beta)$ positive, the latter statements being true as one sees through the fact that y_7 is negative between α and β and has at these points zeros of order $n-1$.

It is clear that the foregoing theorem is best suited to applications to equations of even order. We shall indicate the character of its usefulness by applying it to a special form of the self-adjoint equation of the fourth order, namely,

$$u'''' + \lambda u'' + \lambda^2 \varphi u = 0, \tag{29}$$

where φ is a function of x which is continuous in (ab) and λ is a real parameter independent of x.

As an equation of comparison we take

$$y'''' + 10k^2 y'' + 9k^4 y = 0, \qquad k = \text{const.}, \tag{30}$$

which has the solution $y = \cos^3 k(x-c)$. In order that the theorem shall apply we must have

$$10k^2 = \lambda, \qquad 9k^4 - \lambda^2 \varphi \leqq 0. \tag{31}$$

Therefore we assume that

$$\varphi \geqq \tfrac{9}{100} \text{ in } (ab),$$

so that both conditions (31) are satisfied when $10k^2 = \lambda$. For $x > b$ we take for $\varphi(x)$ the value $\varphi(b)$.

Applying the theorem of this section, we now see that any solution $u(x, \lambda)$ of (29) must have a zero in every interval of length π/k or $\pi\sqrt{10}/\sqrt{\lambda}$ throughout which $x \geqq a$. Hence as λ increases indefinitely the number of roots of $u(x, \lambda)$ in (ab) increases indefinitely.

This example is sufficient to illustrate the nature of the theorem of this section as regards its use in treating oscillation problems for certain interesting special cases of differential equations of any order. It suggests an investigation the carrying out of which the author reserves for a future occasion.

UNIVERSITY OF ILLINOIS,
December, 1916.

NOTE ON A SUBSTITUTE FOR DUHAMEL'S THEOREM.

By Henry B. Fine.

The substitute for Duhamel's Theorem published by G. A. Bliss in the sixteenth volume of this journal* meets all the requirements of the case for students who are mature enough mathematically to understand it. But it is cast in too abstract a form to be serviceable as early in the study of the calculus as some theorem of this kind must be used. A less general theorem is all that is needed, but both in statement and proof it must be sufficiently elementary to be intelligible to a student soon after he has become familiar with the notion of the simple definite integral as the limit of a sum of infinitesimal elements and has acquired some skill in applying this notion. I am therefore led to suggest the following theorem. In content it is the equivalent of a theorem recently published by E. V. Huntington† and like that is a particular case of Bliss's theorem, but the proof, unlike Huntington's and like Bliss's, shows directly the existence of the limit of the sum in question.

Theorem. *Let $f_1(x), f_2(x), \cdots, f_p(x)$ denote any set of functions of x, finite in number, which are continuous in the interval (ab), and let*

$$F(x) = f_1(x)f_2(x)\cdots f_p(x).$$

Suppose the interval (ab) to be divided and redivided into parts in any manner such that as the process is indefinitely continued the greatest of the parts will approach 0 as limit, and at any stage in the process let $h_1, h_2, \cdots, h_n$ represent the parts in length and position; also let $\xi_i', \xi_i'', \cdots, \xi_i^{(p)}$, and ξ_i denote any numbers in the part h_i $(i = 1, 2, \cdots, n)$. Then

$$\lim_{n=\infty} \sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)})h_i = \lim_{n=\infty} \sum_{i=1}^{n} F(\xi_i)h_i = \int_a^b F(x)dx.$$

For any difference of the form $a_1a_2\cdots a_p - b_1b_2\cdots b_p$ can be expressed in terms of the differences $a_i - b_i$ by means of the identity

$$a_1a_2\cdots a_p - b_1b_2\cdots b_p = (a_1 - b_1)a_2a_3\cdots a_p + (a_2 - b_2)b_1a_3\cdots a_p + (a_3 - b_3)b_1b_2a_4\cdots a_p + \cdots + (a_p - b_p)b_1b_2\cdots b_{p-1}.$$

* G. A. Bliss, "A Substitute for Duhamel's Theorem," *Annals of Mathematics*, Ser. 2, vol. 16 (1914).

† E. V. Huntington, "On Setting Up a Definite Integral without the Use of Duhamel's Theorem," The American Mathematical Monthly, vol. 24 (1917).

Applying this identity to the difference

$$f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)}) - f_1(\xi_i)f_2(\xi_i)\cdots f_p(\xi_i)$$

and in the coefficient of each difference $f_j(\xi_i^{(j)}) - f_j(\xi_i)$ on the right replacing every factor f by M, the greatest of the maximum absolute values of the several functions $f_j(x)$ in (ab), we have

$$|f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)}) - F(\xi_i)| \leqq |f_1(\xi_i') - f_1(\xi_i)|M^{p-1} + |f_2(\xi_i'') - f_2(\xi_i)|M^{p-1} + \cdots + |f_p(\xi_i^{(p)}) - f_p(\xi_i)|M^{p-1}.$$

But since all the functions $f_j(x)$ are continuous, and therefore uniformly continuous, in (ab), if any positive number ϵ be assigned we can find an integer n' such that for $n > n'$, and $i = 1, 2, \cdots, n$, we shall have

$$|f_1(\xi_i') - f_1(\xi_i)| < \frac{\epsilon}{pM^{p-1}}, \qquad |f_2(\xi_i'') - f_2(\xi_i)| < \frac{\epsilon}{pM^{p-1}}, \qquad \cdots,$$

$$|f_p(\xi_i^{(p)}) - f_p(\xi_i)| < \frac{\epsilon}{pM^{p-1}},$$

and therefore, by the above inequality,

$$|f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)}) - F(\xi_i)| < \epsilon \qquad (i = 1, 2, \cdots, n).$$

From this it immediately follows, since

$$\sum_{i=1}^{n} h_i = b - a,$$

that

$$\left|\sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)})h_i - \sum_{i=1}^{n} F(\xi_i)h_i\right| < \epsilon(b-a),$$

and therefore, since $\epsilon(b-a)$ may be taken as small as we please, that

$$\lim_{n=\infty}\left\{\sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)})h_i - \sum_{i=1}^{n} F(\xi_i)h_i\right\} = 0.$$

Hence since $\lim_{n=\infty}\sum_{i=1}^{n} F(\xi_i)h_i$ exists, and by definition is $\int_a^b F(x)dx$, the theorem is proved.

The theorem and its proof can be immediately extended to the more general intregal

$$\int_V F(x_1, x_2, \cdots, x_m)dV$$

where F denotes a product of p functions f_j all of which are supposed continuous in the closed region V. It is only necessary in the proof just given to replace h_i by the element ΔV_i of V and to interpret the several ξ_i's as symbols for sets of values of $x_1, x_2, \cdots, x_m$ in ΔV_i.

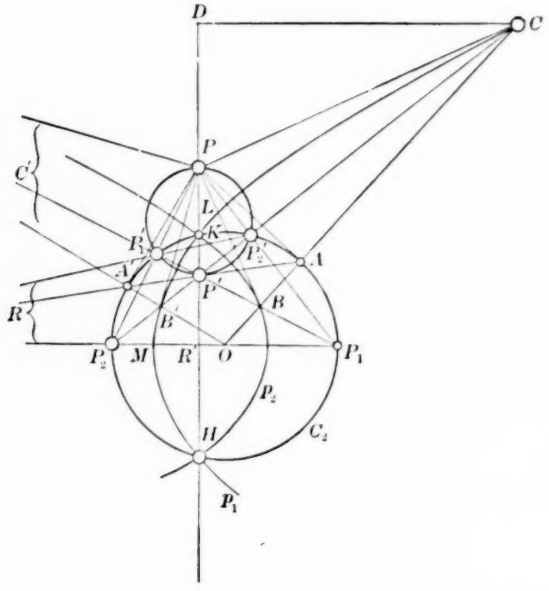
PRINCETON UNIVERSITY.

SOME PROPERTIES OF A STRAIGHT LINE AND CIRCLE AND THEIR ASSOCIATED PARABOLAS.

By James H. Weaver.

In the following article the author sets up a configuration involving circles and confocal parabolas and proves some theorems which he believes to be new.

Let there be a straight line l and a circle C_2 with center O and radius r. The locus of the center of a circle tangent to C_2 and l is two parabolas, with the common focus O, and a common axis, which is the line through O perpendicular to l, and with lines parallel to l and at a distance r from l as directrices. These parabolas may be said to be associated with l and C_2. Call these parabolas p_1 and p_2. Conversely, with any two parabolas having a common axis and focus, there are associated a straight line l and a circle, C_2.



Theorem I: *In the configuration of a straight line l and a circle C_2 let tangents be drawn from any point P on l to C_2 and to the associated parabolas p_1 and p_2. The points of contact of these tangents lie on two straight lines passing through O.*

Proof: Let the points of contact (Fig.) of the tangents from P to C_2 be A and A', of those from P to p_2 be B and C', and of those from P to p_1 be C and B'. Then A, B, and C are collinear with O, and A', B' and C' are collinear with O. For suppose PA drawn from P tangent to C_2. Then draw OA and extend it until it cuts p_1 in some point C. Draw CD perpendicular to l and then draw CP. Since C is on p_1 and CP bisects $\measuredangle DCO$, PC is tangent to p_1 at C. A similar proof holds for the other points of contact.

Theorem II: *The polars of P with respect to p_1 and p_2 intersect C_2 in four points, which determine a quadrangle, whose diagonal points are P, the point P' on l, which is conjugate to P with respect to C_2, and the pole of l with respect to C_2.*

Proof: Let vertex of p_1 be M. Then since $P_2M = MR'$, P_2 is pole of l with respect to p_1. It follows analytically that P_1P_1' and P_2P are perpendicular, and a circle on PP' as diameter passes through P_1' and P_2' and cuts C_2 orthogonally. For if L is the center of the circle on PP', then $\measuredangle LP_2'O$ is right. Because $\measuredangle OP_2'P_2 = \measuredangle OP_2P_2'$ and $\measuredangle LP_2'P_2 = \measuredangle LP'P_2' = \measuredangle P_2P'R'$. But $\measuredangle OP_2P' + \measuredangle P_2P'R' =$ rt $\measuredangle$. $\therefore \measuredangle LP_2'O$ is right. And tangents to C_2 at P_1' and P_2' will intersect on l. $\therefore$ the pole of $P_1'P_2'$ lies on l. It is also evident that the pole of P_1P_2 is the point at infinity on l. Let P_1P_2 and $P_1'P_2'$ intersect in R. Then R is the pole of l with respect to C_2.

The circle on PP' as diameter is one of a system of limit point circles, which have P_1P_2 for radical axis; and R is the radical center of C_2 and any two circles of the system.

From the above discussion the following is evident.

Theorem III: *The perpendiculars from P to the polars of P with respect to p_1 and p_2 intersect these polars on C_2.*

OHIO STATE UNIVERSITY.

MOTIONS IN HYPERSPACE.

By C. L. E. Moore.

1. The literature on motions in ordinary space is quite extensive but that on motions in hyperspace is very limited. The only paper that has a bearing on the subject in hand is one by Cole* in which he studied rotations in 4-space. He found the invariant linear elements to be two completely perpendicular† planes (2-spaces). The method used is entirely unsuited to the study of motions in higher dimensions for the equations increase in complexity as the dimensions increase.

In this note I shall make use of Lie's infinitesimal transformations. Since every finite motion belongs to a one-parameter group, from the invariants of the one-parameter group the invariants of the general motion are obtained. Motions divide themselves into two classes according as the space is even or odd. The general motion in even space is found to be a rotation about a fixed point and leaves invariant $n/2$ mutually completely perpendicular planes. If more than one point is left invariant, an even space will be left invariant. The general motion in odd space has an invariant line and the motion consists of rotations in the hyperplanes perpendicular to this line followed by a translation along the line. If a point is left invariant, an odd space is left fixed point for point.

In this discussion we shall consider only real motions.

2. **The group of infinitesimal transformations.** The group of motions in n-space is a group of $\frac{1}{2}n(n+1)$ parameters and consequently will be generated by $\frac{1}{2}n(n+1)$ independent infinitesimal transformations. The rotations in the coördinate planes (2-spaces) together with the translations along the n coördinate axes form a system of $\frac{1}{2}n(n+1)$ independent infinitesimal transformations. Let us denote this system by

$$(T)\qquad \begin{aligned} X_i(f) &= \frac{\partial f}{\partial x_i}, \qquad i = 1, 2, \cdots, n, \\ X_{ij}(f) &= x_j\frac{\partial f}{\partial x_i} - x_i\frac{\partial f}{\partial x_j} = -X_{ji}, \qquad i, j = 1, 2, \cdots, n. \end{aligned}$$

We then have

$$(X_iX_j) = X_i(X_jf) - X_j(X_if) = 0,$$

$$(X_{ik}X_j) = 0, \quad i, k \neq j, \qquad (X_kX_{jk}) = X_j, \qquad (X_jX_{jk}) = -X_k,$$

$$(X_{ij}X_{kl}) = 0, \quad i, j \neq k, l, \qquad (X_{ij}X_{ik}) = X_{jk}.$$

* On rotations in four dimensions, American Journal of Mathematics, vol. 12, 1889.

† Planes in which any line of one is perpendicular to every line of the other.

That is, the alternants of these transformations are all linearly expressible in terms of them and hence the transformations (T) generate a group of $\frac{1}{2}n(n+1)$ independent parameters. These then will generate our group of motions in n-space.

The general infinitesimal transformation will be a linear combination of the transformations (T).

$$X(f) = \Sigma a_i X_i + \Sigma a_{ij} X_{ij} = \Sigma_i (a_i + \Sigma_k a_{ik} x_k) \frac{\partial f}{\partial x_i},$$

where the a_{ij}'s satisfy the relation $a_{ij} = -a_{ji}$, $a_{jj} = 0$. If this transformation leaves any points fixed

$$a_i + \Sigma_k a_{ik} x_k = 0, \qquad i = 1, 2, \cdots, n. \tag{1}$$

From the above relation on the coefficients a_{ij} we see that the determinant of (1) is skew-symmetric. The rank of a skew-symmetric determinant being even, it is at once seen that the solution of (1) depends first of all on whether n is odd or even.

3. **General motion in even space.** If $n = 2m$, say, the determinant $|a_{ik}|$ in general does not vanish and (1) has a solution, hence one point is left invariant. If this invariant point be taken for the origin the transformations of the group become

$$X = \Sigma a_{ij} X_{ij},$$

and equation (1) becomes

$$\Sigma a_{ik} x_k = 0.$$

If $|a_{ik}|$ is of rank $2m$ the origin is the only invariant point. If the rank is $2m - 2$ there is a plane of invariant points and in general a linear space of even dimensions is always left invariant. When the rank is 2 a space of $2m - 2$ dimensions is left invariant point for point. In this case the motion is a rotation parallel to a fixed plane and is therefore simply isomorphic with the rotations in a plane.

In the general case only a single point is left fixed. We shall now examine whether or not there are any linear spaces which are left invariant as a whole (the individual points not being left invariant). Let us see first if any hyperplanes passing through the fixed point O are left invariant. Since the origin is left fixed the transformation belongs to the group of homogeneous transformations. The equation of a hyperplane passing through O is

$$\Sigma \lambda_k x_k = 0,$$

and the condition that this be left invariant is

$$\Sigma_i \Sigma_k \lambda_i a_{ik} x_k = \rho \Sigma_k \lambda_k x_k,$$

that is

$$\Sigma_i \lambda_i a_{ik} = \rho \lambda_k. \tag{2}$$

This being homogeneous in λ will have a solution only when the determinant of the system vanishes. This determinant is

$$\begin{vmatrix} -\rho & a_{12} & a_{13} & \cdots & a_{1,2m} \\ -a_{12} & -\rho & a_{23} & \cdots & a_{2,2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{1,2m} & \cdot & \cdot & \cdot & -\rho \end{vmatrix}$$

and is seen to be a skew determinant. If such a determinant is expanded in powers of the diagonal terms, the coefficients of the various powers of ρ will be sums of principal minors of $|a_{ik}|$. That is, in the equation

$$\rho^{2m} + A_1\rho^{2m-1} + A_2\rho^{2m-2} + \cdots A_n = 0 \tag{3}$$

$A_1 = 0$, A_2 is the sum of all the principal minors of $|a_{ik}|$ of order two (that is, minors whose principal diagonals are elements of the principal diagonal of $|a_{ik}|$), A_3 is the sum of all the principal minors of order three, etc. Thus all the coefficients are sums of skew-symmetric determinants. But such determinants of odd order vanish and those of even order are perfect squares. Hence in (3) there are no odd powers of ρ and the coefficients of the even powers are all positive. Therefore (3) has m pairs of conjugate imaginary roots. *There are then m pairs of conjugate imaginary hyperplanes left invariant by a general real motion in a space of $2m$ dimensions.* A pair of conjugate imaginary hyperplanes will intersect in a real space of $2m - 2$ dimensions, a pair of these will intersect in a real space of $2m - 4$ dimensions and finally $m - 1$ of these will intersect in a real plane. Hence a motion in a space of $2m$ dimensions leaves m $(2m - 2)$-dimensional spaces invariant, $\frac{1}{2}m(m - 1)$ spaces of $2m - 4$ dimensions, and finally m planes invariant. All these spaces pass through the origin and are real.

If a plane is left invariant, then the S_{2m-2} perpendicular to it and passing through O must also be left invariant. For the S_{2m-2} is generated by the lines passing through O and perpendicular to the fixed plane and since the origin is left fixed this set of lines will go into itself and hence the space generated by them is left invariant. In this S_{2m-2} a plane is left invariant and these two planes are completely perpendicular. In the S_{2m-4} lying the S_{2m-2} and perpendicular to the second plane, a plane is also invariant which is completely perpendicular to the other two and so on. Hence *the general motion in $2m$ dimensions leaves invariant m completely perpendicular planes.*

There are $2m$ minimal hyperplanes left invariant. For the minimal cone $\Sigma x_i^2 = 0$ is left invariant and hence tangent planes to it go into tangent planes. Through each invariant S_{2m-2} pass two of these tangent hyper-

planes and as the motion is an infinitesimal one these must be left invariant. Hence the $2m$ invariant hyperplanes must be minimal hyperplanes.

If we take these invariant planes as the coördinate planes the transformation (T) takes the form

$$x_2\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial x_2}, \qquad x_4\frac{\partial f}{\partial x_3} - x_3\frac{\partial f}{\partial x_4}, \qquad x_6\frac{\partial f}{\partial x_5} - x_5\frac{\partial f}{\partial x_6}, \qquad \cdots,$$

$$x_{2m}\frac{\partial f}{\partial x_{2m-1}} - x_{2m-1}\frac{\partial f}{\partial x_{2m}}.$$

Taking this form for (T) we see that the path curves of the group are the solutions of the set of equations

$$a_1\frac{dx_1}{x_2} = -a_1\frac{dx_2}{x_1} = a_2\frac{dx_3}{x_4} = -a_2\frac{dx_4}{x_3} = \cdots = a_m\frac{dx_{2m-1}}{x_{2m}} = -a_m\frac{dx_{2m}}{x_{2m-2}}.$$

Hence the path curves lie on the intersection of the cylinders

$$x_1^2 + x_2^2 = \bar{x}_1^2 + \bar{x}_2^2, \qquad x_3^2 + x_4^2 = \bar{x}_3^2 + \bar{x}_4^2, \qquad \cdots,$$

$$x_{2m-1}^2 + x_{2m}^2 = \bar{x}_{2m-1}^2 + \bar{x}_{2m}^2,$$

where $\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots, \bar{x}_{2m}$ are the initial coördinates of the point. Hence we see that the projection of the path curves on each of the invariant planes is a circle. This fact is evident geometrically for if the point P (not in a fixed plane) projects on one of the invariant planes into the point Q, then the triangle OPQ by the group will be so moved that O remains fixed, while Q will always remain in the fixed plane. The distance OQ is left invariant and hence Q will describe a circle. The locus of P is not a circle, for this would mean that the plane of the circle is invariant, which, in general, is impossible.

If $|a_{ik}| = 0$, we see from equation (3) that two values of ρ are zero. In this case it is seen from (2) that a one-parameter family of real hyperplanes is left invariant, since the rank of $|a_{ik}|$ is now $2m - 2$. We saw that this was the condition for a plane of invariant points. The invariant hyperplanes are then those containing the S_{2m-2} perpendicular to this absolutely invariant plane. This is evident since in this case the same equations that determine the coördinates of the hyperplane also determine the coördinates of the invariant points. The coördinates of any point P in the fixed plane are proportional to the direction cosines of the hyperplane perpendicular to OP. Now if these coördinates are the same as the coördinates of one of these hyperplanes then this hyperplane must be perpendicular to OP. Hence the invariant hyperplanes contain the S_{2m-2} normal to the fixed plane.

In this case there are still m invariant planes but one of them is left absolutely invariant. In the same way we see that if the rank of $|a_{ik}|$ is $2m - 4$, there is a three-parameter family of real invariant hyperplanes, and two of the invariant planes are point for point invariant. The same reasoning holds when the rank of $|a_{ik}|$ is reduced two at a time. If the rank is less than 2 the motion reduces to an identity. If $m - 1$ of the fixed planes are absolutely invariant (X), becomes the group of motions in a plane; if $m - 2$ are absolutely invariant, the group becomes the rotations in S_4 and so on.

4. **Motions in even space which have no fixed points.** We have seen that, in general, in a space of even dimensions one point at least is always left fixed. We know, however, that in the plane there are motions which have no fixed points, viz., translations. We shall now examine what are the conditions that the same thing happen in higher dimensions. Equation (1), if $|a_{ik}| = 0$, are inconsistent except for particular values of a_i. In case they are inconsistent we may say that the invariant point has moved off to infinity. This would then indicate that there is an invariant plane in which a pencil of parallel lines and consequently every pencil of parallel lines are left invariant. We will now show that this is the case. There are ∞^1 directions left invariant. For the coefficients of direction of the line joining two points x, y are $x_i - y_i$ and if we write the equations of the motion in the form

$$x_i' = x_i + (a_i + \Sigma a_{ij}x_j)dt \tag{4}$$

we have at once the relation

$$x_i' - y_i' = x_i - y_i + \Sigma a_{ij}(x_j - y_j)dt.$$

The direction $x_j - y_j$ will then be left invariant if

$$\Sigma a_{ij}(x_j - y_j) = 0. \tag{5}$$

But since we assumed that $|a_{ij}| = 0$ and consequently its first minors, it follows that there are at least ∞^1 sets of values of $x_j - y_j$, not all zero, which will satisfy (5) and which are therefore left invariant by the motion.

We can now show that there is one pencil of these lines in which the individual lines are left invariant, that is, the points of one of those lines move along the line itself. A plane containing two of these invariant directions u_i, u_i' can be written

$$x_i = \bar{x}_i + u_i d\xi + u_i' d\eta. \tag{6}$$

Since u_i, u_i' are homogeneous we can choose them so that they will satisfy the relations

$$\Sigma u_i^2 = 1, \qquad \Sigma u_i'^2 = 1, \qquad \Sigma u_i u_i' = 0.$$

The first two say that the u's are actual direction cosines and the third that the two directions are perpendicular. Now by (4) $\bar{x}_i$ is transformed into $\bar{x}_i + (a_i + \Sigma a_{ij}\bar{x}_j)dt$ and if this new point satisfies (6) we must be able to find values of $d\xi/dt$, $d\eta/dt$ which will render the system of equations

$$\Sigma a_{ij}x_j + a_i = u_i\frac{d\xi}{dt} + u_i'\frac{d\eta}{dt} \tag{7}$$

consistent. The determinant $|a_{ij}|$ of this system of equations is of rank $2m-2$ or lower, and therefore if they are consistent there must be two linear relations connecting them. Multiplying by u_i and summing on i and then on u_i' and summing on i, we have

$$\Sigma a_i u_i = \frac{d\xi}{dt}, \qquad \Sigma a_i u_i' = \frac{d\eta}{dt}$$

by virtue of (5) and the relation between the u's. We then have a set of values of $d\xi/dt$ and $d\eta/dt$ which will render the set of equations (7) consistent. The values of $\bar{x}_i$ will then be the solutions of (7) after we have replaced $d\xi/dt$, $d\eta/dt$ by the values found above. But in (7) there are only $2m-2$ independent equations and therefore there is a plane of these solutions. Hence we have a plane which is invariant and which contains a pencil of invariant parallel lines. In this plane any line is transformed into a parallel line and consequently in it the transformation is a translation.

An S_{2m-2} normal to the plane just found will cut it in a single point. By (6) this point is translated along the invariant line passing through it and the S_{2m-2} itself is rotated parallel to itself. This motion then is simply isomorphic to the general motion in an S_{2m-1} determined by the S_{2m-2} and the invariant line cutting it. The motions in odd space will be taken up later. If we take the origin in this plane and the x_{2m-1}-axis as one of the invariant lines and the line in the plane perpendicular to it as the x_{2m}-axis, then the transformations of such a motion can be written

$$x_2\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}, \; x_4\frac{\partial f}{\partial x_3} - x_3\frac{\partial f}{\partial x_4}, \; \cdots, \; x_{2m-3}\frac{\partial f}{\partial x_{2m-2}} - x_{2m-2}\frac{\partial f}{\partial x_{2m-3}}, \; \frac{\partial f}{\partial x_{2m-1}}.$$

We shall see later that these are identical with the equations for the general motion in odd space.

The equation of a hyperplane which passes through the fixed plane is

$$\sum_1^{2m-2}\lambda_i x_i = 0.$$

If this hyperplane is left invariant by (4), it would be necessary that it be

left invariant by the rotation

$$x_i' = x_i + \Sigma a_{ij}x_j dt, \qquad i, j = 1, 2, \cdots, 2m - 2.$$

This then becomes identical with the problem previously solved, that is, that represented by equation (2). There are then $m - 1$ pairs of conjugate imaginary hyperplanes left invariant by (4). By reasoning similar to that used before we see that there are $m - 1$ spaces of four dimensions left invariant. These are the four-spaces determined by the invariant plane and any one of the $m - 1$ planes left invariant in an S_{2m-2} normal to the invariant plane by the rotation part of the group. That is, we consider the motion to be broken up into a rotation followed by a translation. Within one of these invariant four-spaces there is also an infinite number of three-spaces left invariant. They will be those determined by an invariant plane in the S_{2m-2} and the invariant line which cuts it.

It is easy to see what the invariant configurations are in case $|a_{ij}|$ is of lower rank.

5. **Motions in odd space.** If n is odd the condition for invariant points is again given by equation (1). But in this case the determinant being skew-symmetric of odd order must vanish and hence (1) is an inconsistent set of equations except for particular values of a_i. By processes similar to those previously used we see that the condition that the direction of a line joining two points x, y be invariant is

$$\Sigma a_{ij}u_j = 0, \qquad i = 1, 2, \cdots, n,$$

where $u_i = x_i - y_i$. Since $|a_{ij}| = 0$, it follows that there is always one invariant direction. Since the u's are a homogeneous set, we can determine them so that $\Sigma u_i^2 = 1$. This of course excludes minimal lines but since we are interested only in real motions the coefficients a_{ij} are real and consequently the u's are real and the minimal lines are excluded.

If the rank of $|a_{ij}|$ is $n - 1$, there is just one invariant line. To show this we proceed the same as for the case when n was even. A line having this direction may be written

$$x_i = \bar{x}_i + u_i d\tau.$$

Now by (1) $\bar{x}_i$ is transformed into $\bar{x}_i + (a_i + \Sigma a_{ij}\bar{x}_j)dt$ and, if this point lies on the line, we have

$$a_i + \Sigma a_{ij}\bar{x}_j = u_i \frac{d\tau}{dt}.$$

Now it must be possible to find a value of τ which will render this system

consistent. This will be so when a linear combination of them vanishes. Multiply each equation by u_i and sum, then

$$\tau = \Sigma a_i u_i.$$

Substituting this value of τ, we have the invariant line determined as the intersection of the hyperplanes

$$\Sigma a_{ij} x_j + a_i = u_i \frac{d\tau}{dt}.$$

There are ∞^1 solutions of this set of equations since the determinant is of rank $n - 1$.

If the values of a_i are such as to render equations (1) consistent, then there will be a line of invariant points, that is the invariant line has become fixed point for point. In this case the motion reduces to a rotation about the fixed line and is simply isomorphic to the rotations in the even space S_{n-1} perpendicular to the axis of rotation. We may say then if $|a_{ij}|$ is of rank $n - 1$, there is either no fixed point or a line of fixed points.

If the rank of $|a_{ij}|$ reduces to $n - 3$, then by continuing the same argument as above we see that there are ∞^2 directions left invariant and that ∞^2 lines of an S_3 having this direction are left invariant. The motion then consists of a rotation in an S_{n-3} perpendicular to the S_3 followed by a translation along the invariant line which cuts it. In this case if a_i are such as to render the system of equations consistent, then the S_3 will be left invariant point for point. The other cases follow the same arguments. We may then say in general, if no point is left invariant, then an odd space generated by a system of parallel invariant lines is left invariant. If one point is left invariant, then an odd space is left absolutely invariant.

In the general case if the axis x_n is taken as the fixed line, the infinitesimal transformations of the group become

$$x_i' = x_i + \Sigma a_{ij} x_j dt, \qquad i, j = 1, 2, \cdots, n - 1,$$

$$x_n' = x_n + a_n dt.$$

The equation of a hyperplane which contains the invariant line is

$$\sum_i^{n-1} \lambda_i x_i = 0,$$

and if it is left invariant,

$$\Sigma\Sigma \lambda_i a_{ij} x_j = \rho \Sigma \lambda_i x_i.$$

This is the same as equation (2) and hence there are $(n-1)/2$ invariant conjugate imaginary hyperplanes and consequently $(n-1)/2$ real invariant S_3's. Again this could have been inferred from the fact that in an S_{n-1} perpendicular to x_n there are $(n-1)/2$ invariant planes and an S_3 determined by each of these planes and x_n is left invariant. The cases in which the rank of $|a_{ij}|$ is less than $n-1$ are easily handled as before.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

A DEFINITION OF SENSE ON CLOSED CURVES IN NON-METRICAL PLANE ANALYSIS SITUS.*

By John Robert Kline.

1. Introduction.

One of the main aims of research in the foundations of geometry is to free geometry from any dependence upon the intuition. It is the purpose of the present paper to give a rigorous definition of a notion, which is very frequently left to the intuition—the notion of sameness of sense† on closed curves.‡ All the theorems of the present paper are proved on the basis of Professor R. L. Moore's system of axioms Σ_3.§ *Hence all the results hold even in certain spaces which are neither metrical, descriptive nor separable.*

The definition proposed is as follows: *The sense $A_1B_1C_1$ on the closed curve J_1 and the sense $A_2B_2C_2$ on the closed curve‖ J_2 are said to be the same with respect to their common exterior E_{12}, if there exists in E_{12} a simple closed curve J_3 and three distinct points A_3, B_3 and C_3 thereon such that* (1) *if it is impossible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 by arcs, no two of which have a point in common and which lie except for their endpoints in E_{13} (the common exterior of J_1 and J_3), then it is also impossible to join*

* Presented to the Society, April 29, 1916.

† I wish to thank Professor R. L. Moore, who suggested the undertaking of this investigation. The general definition given in this paper for sameness of sense on any two closed curves is a modification of the following definition suggested by him for the case where the closed curves have no point in common: If the distinct points A_i, B_i and C_i $(i = 1, 2)$ of the non-intersecting closed curves J_1 and J_2 are such that there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two of which have a point in common and which have, except for their endpoints, no point in common with either J_1 or J_2, then the sense $A_1B_1C_1$ on J_1 is said to be *opposite* to the sense $A_2B_2C_2$ on J_2 with respect to the region in which these arcs are drawn. If such a joining is not possible then the senses are said to be the *same*.

‡ If A and B are distinct points, then a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B but containing no proper connected subset containing both A and B. See N. J. Lennes, Curves in non-metrical analysis situs with an application in the calculus of variations, American Journal of Mathematics, vol. 33 (1911), p. 308. In the present paper "arc" and "simple continuous arc" will be considered synonymous terms. A *simple closed curve* is a set of points composed of two arcs AXB and AYB which have no point in common other than A and B.

§ Cf. R. L. Moore, On the foundations of plane analysis situs, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–64.

‖ Hereafter in this paper, "closed curve" and "simple closed curve" will be used synonymously. In the present definition A_i, B_i and C_i $(i = 1, 2, 3)$ are distinct points of the closed curve J_i.

A_2 to A_3, B_2 to B_3 and C_2 to C_3 by arcs, no two of which have a point in common and which lie except for their endpoints in E_{23} (the common exterior of J_2 and J_3), (2) if it is possible to join A_1 to A_3, B_1 to B_3 and C_1 to C_3 as indicated above, then it is also possible to join A_2 to A_3, B_2 to B_3 and C_2 to C_3 as indicated. Otherwise the sense $A_1B_1C_1$ on J_1 and the sense $A_2B_2C_2$ on J_2 are said to be opposite with respect to E_{12}.

2. Certain Theorems in Analysis Situs.

Before establishing certain properties of sense, we shall first prove a number of preliminary theorems in analysis situs.

Theorem A: *Suppose the arc $A_1X_2A_2$ is such that $\overset{\frown}{A_1X_2A_2}$* has no point in common with the arcs $A_1B_1C_1$ and $A_2B_2C_2$ of the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there are on the arc $A_1X_2A_2$ at most a finite number of distinct subarcs $K_iP_iL_i$ such that (1) K_i is a point of A_1XA_2 while L_i is a point of C_1ZC_2, (2) no point of $\overset{\frown}{K_iP_iL_i}$ is on $A_1B_1C_1ZC_2B_2A_2XA_1$.*

Proof: Suppose Theorem A is false. Then there exists at least one arc $A_1X_1A_2$ such that (1) $\overset{\frown}{A_1X_1A_2}$ has no point in common with either the arc $A_1B_1C_1$ or the arc $\overset{\frown}{A_2B_2C_2}$ of $A_1B_1C_1ZC_2B_2A_2XA_1$, (2) there are on $A_1X_1A_2$ infinitely many distinct points $[K]$ such that (a) to each point K' of $[K]$ there corresponds a point $L_{k'}$, common to $A_1X_1A_2$ and C_1ZC_2 and an arc $K'P_{k'}L_{k'}$, which is a subset of $A_1X_1A_2$, (b) each point of $[K]$ is on A_1XA_2, (c) $\overset{\frown}{K'P_{k'}L_{k'}}$ has no point in common with the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$, (d) the order $A_1K'L_{k'}A_2$ holds on $A_1X_1A_2$. As the point set $[K]$ is a subset of $A_1X_1A_2$, it is bounded. Hence $[K]$ has at least one limit point $\bar{K}$.† Hence there exists a sequence of points K_1, K_2, K_3, $\cdots$, such that (1) each point of the sequence belongs to $[K]$, (2) $\bar{K}$ is the only limit point of K_1, K_2, K_3, $\cdots$,‡ (3) either the order $A_1K_1K_2K_3\cdots\bar{K}A_2$ or the order $A_1\bar{K}\cdots K_3K_2K_1A_2$ holds on A_1XA_2. To every point K_i there corresponds a point L_i, common to $A_1X_1A_2$ and C_1ZC_2 and an arc $K_iP_iL_i$, such that $\overset{\frown}{K_iP_iL_i}$ is a subset of $A_1X_1A_2$ and has no point in common with $A_1B_1C_1\overset{\frown}{ZC_2}B_2A_2XA_1$. For all values of i the order $A_1K_iL_iA_2$ holds on $A_1X_1A_2$. As the arc $K_iP_iL_i$ and the arc

* If AXB is an arc, then $\overset{\frown}{AXB}$ denotes the point set $AXB - A - B$. Likewise if AB is an arc, then $\overset{\frown}{AB}$ denotes the point set $AB - A - B$.

† Cf. F. Hausdorff, Grundzüge der Mengenlehre, Veit and Co., Leipzig, 1914, p. 231. See also E. W. Chittenden, The Heine Borel Theorem in a Riesz Domain, Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 179–83, and vol. 20 (1914), p. 461. For a proof that, in the presence of certain linear order axioms, the Heine Borel Theorem is equivalent to the Dedekind Cut Postulate, see O. Veblen, The Heine-Borel Theorem, Bulletin of the American Mathematical Society, vol. 10 (1903–4), pp. 463–9.

‡ Cf. R. L. Moore, loc. cit., Theorems 8 and 9, p. 134.

$K_jP_jL_j$ $(i \neq j)$ have no point in common, either the order $A_1K_1L_1K_2L_2\cdots KA_2$ or the order $A_1\bar{K}\cdots K_2L_2K_1L_1A_2$ holds on $A_1X_1A_2$.

Case I. Suppose $A_1K_1L_1K_2L_2\cdots\bar{K}A_2$ holds. It may easily be shown that $\bar{K}$ is a limit point of the set $L_1, L_2, L_3, \cdots$. As every point of $L_1, L_2, L_3, \cdots$, belongs to C_1ZC_2, a closed point set, $\bar{K}$ belongs to C_1ZC_2. But $\bar{K}$ is on A_1XA_2. Hence C_1ZC_2 and A_1XA_2 have at least one point in common, contrary to hypothesis. Thus, in Case I, we are led to a contradiction.

Case II. Suppose $A_1\bar{K}\cdots K_2L_2K_1L_1A_2$ holds on $A_1X_1A_2$. This is impossible as in Case I.

It may easily be seen that if $A_1X_1A_2$ has any point in common with C_1ZC_2, then there exist $2n$ distinct subarcs of $A_1X_1A_2$, $K_1P_1L_1$, $L_2P_2K_2$, $\cdots$, $L_{2n}P_{2n}K_{2n}$, where $K_1, K_2, \cdots, K_{2n}$ is a subset of A_1XA_2 while $L_1, L_2, \cdots, L_{2n}$ is a subset of C_1ZC_2 and the set $\overset{\frown}{K_iP_iL_i}$ $(i = 1, 2, \cdots, 2n)$ has no point in common with $A_1B_1C_1ZC_2B_2A_2XA_1$. On $A_1X_1A_2$ the order $K_1L_1K_2L_2K_4\cdots L_{2n-1}K_{2n}$ holds. If $K_1 \neq A_1$, then there is on the subarc A_1K_1 of $A_1X_1A_2$ no point of C_1ZC_2. Likewise if $K_{2n} \neq A_2$, there is on the subarc $K_{2n}A_2$ of $A_1X_1A_2$ no point of C_1ZC_2. If $K_{2i+1} \neq K_{2i}$ $(i = 1, 2, \cdots, n-1)$ then K_{2i+1} is a point of the subset $\overset{\frown}{K_{2i}L_{2i+1}}$ of $A_1X_1A_2$. If $L_{2i} \neq L_{2i-1}$ $(i = 1, 2, \cdots, n)$, then L_{2i} is a point of the subset $\overset{\frown}{L_{2i-1}K_{2i}}$ of $A_1X_1A_2$.

Theorem B: *Suppose the closed curves J_1 and J_2 are such that $J_1 + I_1$* is a subset of E_2, (2) A_i, B_i and C_i are distinct points of J_i, (3) A_1XA_2 and C_1ZC_2 are arcs which have no point in common and lie except for their endpoints in E_{12}. Under these conditions, in order that there exist an arc B_1YB_2, which fails to meet either A_1XA_2 or C_1ZC_2 and lies except for its endpoints in E_{12}, it is both necessary and sufficient that I_1 and I_2 be either both within or both without $A_1B_1C_1ZC_2B_2A_2XA_1$.*

Proof: *The condition is sufficient.* Suppose I_1 and I_2 are both within $A_1B_1C_1ZC_2B_2A_2XA_1$.† Let D_i $(i = 1, 2)$ denote a point of J_i such that A_i and C_i separate B_i and D_i on J_i. All points of $\overset{\frown}{A_iD_iC_i}$ are within $A_1B_1C_1ZC_2B_2A_2XA_1$. For suppose some point P of $\overset{\frown}{A_1D_1C_1}$ were without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exists a region R containing P and lying entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$. Hence R contains no point of I_1, which is entirely within $A_1B_1C_1ZC_2B_2A_2XA_1$. But every region

* If J_i is a closed curve, I_i will denote the interior of J_i while E_i denotes its exterior. The symbol E_{ij} will denote the common exterior of J_i and J_j.

† That I_i $(i = 1, 2)$ lies either entirely within or entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$ follows at once from the following facts: (1) a region is a connected point set; (2) no point $\{^{\text{within}}_{\text{without}}\}$ a region is a limit point of a set of points lying entirely $\{^{\text{without}}_{\text{within}}\}$ that region.

containing a point of J_i must contain a point of I_i. Hence the supposition that a point of $\overset{\frown}{A_1D_1C_1}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$ leads to a contradiction. In like manner all points of $\overset{\frown}{A_2D_2C_2}$ are within $A_1B_1C_1ZC_2B_2A_2XA_1$. Let M denote any point without this closed curve. Join M to B_1 and M to B_2 by arcs MB_1 and MB_2, lying except for B_1 and B_2, respectively, entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$.* The point set $MB_1 + MB_2$ will contain as a subset an arc B_1YB_2, which has no point in common with $A_1XA_2 + C_1ZC_2$ and lies, except for its endpoints, entirely in E_{12}. In case I_1 and I_2 are both without $A_1B_1C_1ZC_2B_2A_2XA_1$, it follows that $\overset{\frown}{A_1D_1C_1} + \overset{\frown}{A_2D_2C_2}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exists an arc B_1YB_2 such that $\overset{\frown}{B_1YB_2}$ is a subset of the interior of $A_1B_1C_1ZC_2B_2A_2XA_1$.

(*b*) *The condition is necessary.* Suppose there is an arc B_1YB_2, which fails to meet either A_1XA_2 or C_1ZC_2 and such that $\overset{\frown}{B_1YB_2}$ lies in E_{12}, while I_1 is within and I_2 is without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then it may easily be proved that $\overset{\frown}{A_1D_1C_1}$ is within and $\overset{\frown}{A_2D_2C_2}$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$ while $\overset{\frown}{B_1YB_2}$ is either entirely within or entirely without this closed curve. But if $\overset{\frown}{B_1YB_2}$ is entirely within $A_1B_1C_1ZC_2B_2A_2XA_1$, then it has at least one point in common with $\overset{\frown}{A_1D_1C_1}$ while if it lies entirely without $A_1B_1C_1ZC_2B_2A_2XA_1$, it has at least one point in common with $\overset{\frown}{A_2D_2C_2}$.† But this is contrary to our assumption. In like manner, if we suppose I_1 is without and I_2 is within $A_1B_1C_1ZC_2B_2A_2XA_1$, we are led to a contradiction.

Theorem C: *Suppose the closed curve J_1 and its interior I_1 are without the closed curve J_2, (2) A_i and C_i separate B_i and D_i on $J_i (i=1, 2)$, (3) there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two of which have a point in common and which lie except for their endpoints in E_{12}. Under these conditions, there exists an arc D_1WD_2, which fails to meet $A_1XA_2 + C_1ZC_2$ and lies except for its endpoints in E_{12}.*

Theorem D: *If the points A and C separate B and D on the closed curve J while K is a closed point set such that K has no point in common with $\overset{\frown}{ABC}$, then there exists an arc AFC such that $\overset{\frown}{AFC}$ is within J and fails to meet K.*

Proof: Put about B a closed curve $\bar{C}$ such that neither $\bar{C}$ nor its interior contains a point of $ADC + K$. Let C_1 be the first point of the subarc BC of ABC which is on $\bar{C}$ while A_1 is the first point of the subarc BA of

* Cf. R. L. Moore, loc. cit., Theorem 39, pp. 153–5.

† Cf. R. L. Moore, loc. cit., Theorem 29, pp. 145–6.

CBA on $\bar{C}$. There exists a closed curve J_1 such that (1) B is on J_1, (2) all points on J_1 are points either of J or of $\bar{C}$, (3) every point within J_1 is within J and within $\bar{C}$.* All points of A_1BC_1 are on J_1. There exists an arc $A_1X_1C_1$ such that $\overset{\frown}{A_1X_1C_1}$ is within J_1 and hence within J and fails to meet either ADC or K. Let Q be a point within $A_1BC_1X_1A_1$.

If in the proof of Theorem 32† of Prof. Moore's paper we insist that for all values of i greater than 1, the interior of $B_iX_iC_iC_{i-1}B_i$‡ fails to contain any point of $K + Q + C$, while (1) $B_iX_iC_i$ is within J and fails to meet K, (2) $C_iC_{i-1}B_i$ is a subset of the arc BC of ABC, we obtain a proof that there exists an arc QC such that $\overset{\frown}{QC} + Q$ is a subset of the interior of J and fails to meet K. In the same manner we may prove that there exists an arc QA such that $\overset{\frown}{QA} + Q$ is a subset of the interior of J and fails to meet K. The point set $QC + QA$ contains as a subset an arc AQ_1C, such that $\overset{\frown}{AQ_1C}$ is within J and fails to meet K.

A similar proof will show that there exists an arc AGC such that $\overset{\frown}{AGC}$ is without J and fails to meet K.

Theorem E: *Suppose* (1) *the closed curve* J_1 *and its interior* I_1 *are without the closed curve* J_2, (2) A_i *and* C_i $(i = 1, 2)$ *separate* B_i *and* D_i *on* J_i, (3) $A_1X_1A_2$, $A_1X_2A_2$ *and* $C_1Z_1C_2$ *are arcs lying except for their endpoints in* E_{12} *and such that neither* $A_1X_1A_2$ *nor* $A_1X_2A_2$ *meets* $C_1Z_1C_2$, (4) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. *Then* I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

Proof: Case I. I_1 and I_2 are both without $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then it follows that $\overset{\frown}{A_1D_1C_1}$ and $\overset{\frown}{A_2D_2C_2}$ are both without $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. There are three possibilities:

(*a*) $\overset{\frown}{A_1X_2A_2}$ is a subset of the interior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then the interior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1 = \overset{\frown}{A_1X_2A_2}$ + the interior of $A_1X_1A_2X_2A_1$ + the interior of $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$. Hence I_1 and I_2 are both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

(*b*) $\overset{\frown}{A_1X_2A_2}$ is a subset of the exterior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. By Theorem B, there exists an arc B_1YB_2 such that $\overset{\frown}{B_1YB_2}$ is a subset of the interior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then, as B_1YB_2 has no point in common with either $A_1X_2A_2$ or $C_1Z_1C_2$ while $\overset{\frown}{B_1YB_2}$ lies in E_{12}, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

(*c*) $\overset{\frown}{A_1X_2A_2}$ is not a subset either of the interior or of the exterior of $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then, by Theorem D, there exists an arc A_1EA_2

* Cf. R. L. Moore, loc. cit., Theorem 43, pp. 156–7.

† Cf. R. L. Moore, loc. cit., Theorem 32, pp. 147–8.

‡ B_i corresponds to A_i and C_i to B_i in Professor Moore's notation.

such that A_1EA_2 is within $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$ and fails to meet $A_1X_2A_2$. By (a), I_1 and I_2 are both without $A_1B_1C_1Z_1C_2B_2A_2EA_1$. By ($b$) I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

Case II. I_1 and I_2 are both within $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. Then I_1 and I_2 are both without $A_1D_1C_1Z_1C_2D_2A_2X_1A_1$. By Case I, I_1 and I_2 are either both within or both without $A_1D_1C_1Z_1C_2D_2A_2X_2A_1$. Then, by Theorems B and C, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_2A_1$.

Theorem F: *Suppose* (1) *the closed curve* J_1 *and its interior* I_1 *are without the closed curve* J_2, (2) A_i *and* C_i ($i = 1, 2$) *separate* B_i *and* D_i *on* J_i, (3) $A_1X_1A_2$, $A_1X_2A_2$ *and* $C_1Z_1C_2$ *are arcs lying except for their endpoints in* E_{12}, *while* $A_1X_1A_2$ *and* $C_1Z_1C_2$ *have no points in common*, (4) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. *Under these conditions there exist arcs* A_1XA_2 *and* C_1ZC_2 *such that* (1) A_1XA_2 *and* C_1ZC_2 *have no points in common and lie except for their endpoints in* E_{12}, (2) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1ZC_2B_2A_2XA_1$, (3) *any point of* $A_1X_2A_2$, *which is not on* $A_1B_1C_1ZC_2B_2A_2XA_1$, *is a point of one of* $2n$ *distinct subarcs of* $A_1X_2A_2$, $K_iP_iL_i$ ($i = 1, 2, \cdots, 2n$) *where* K_i *is a point common to* $A_1X_2A_2$ *and* A_1XA_2, L_i *is a point common to* $A_1X_2A_2$ *and* C_1ZC_2, *while no point of* $K_iP_iL_i$ *is on* $A_1B_1C_1ZC_2B_2A_2XA_1$, (4) *on* $A_1X_2A_2$, *the order* $K_1L_1K_2L_3\cdots K_{2n}$ *holds, while if* $K_{2j+1} \neq K_{2j}$ ($j = 1, 2, \cdots, n-1$) *then* K_{2j+1} *is a point of the subset* $K_{2j}L_{2j+1}$ *on* $A_1X_2A_2$ *and if* $L_{2m} \neq L_{2m-1}$ ($m = 1, 2, \cdots, n$) *then* L_{2m} *is a point of the subset* $L_{2m-1}K_{2m}$ *on* $A_1X_2A_2$, (5) *if* $K_1 \neq A_1$ *then the arc* A_1K_1 *of* A_1XA_2 *is the arc* K_1A_1 *of* $A_1X_2A_2$ *and if* $K_{2n} \neq A_2$ *then the arc* $K_{2n}A_2$ *on* A_1XA_2 *is the arc* $K_{2n}A_2$ *of* $A_1X_2A_2$, (6) *on* A_1XA_2 *the order* $K_1K_2K_4\cdots K_{2n}$ *holds while if* $K_{2j+1} \neq K_{2j}$, *then the arc* $K_{2j}K_{2j+1}$ *of* A_1XA_2 *is the arc* $K_{2j}K_{2j+1}$ *of* $A_1X_2A_2$ *while* K_{2j+1} *is a point of* $K_{2j}K_{2(j+1)}$ *on* A_1XA_2, (7) *on* C_1ZC_2 *the order* $C_1L_1L_3\cdots L_{2n-1}C_2$ *holds while if* $L_{2m} \neq L_{2m-1}$, *then* L_{2m} *is a point of the subset* $L_{2m-1}L_{2m+1}$ *of* C_1ZC_2* *and the arc* $L_{2m-1}L_{2m}$ *of* C_1ZC_2 *is the arc* $L_{2m-1}L_{2m}$ *of* $A_1X_2A_2$.

Proof: Let L_1 denote the first point which $A_1X_2A_2$ has in common with $C_1Z_1C_2$. [If no such point exists, then $A_1X_2A_2$ fails to meet $C_1Z_1C_2$. Then by Theorem E, $A_1X_2A_2$ may be taken as A_1XA_2 while $C_1Z_1C_2$ is C_1ZC_2.] Let K_1 denote the last point which the arc $A_1X_1A_2$ has in common with the arc A_1L_1 of $A_1X_2A_2$ (Fig. 1). Let $A_1X_{11}A_2$ denote the arc composed of A_1K_1 (on $A_1X_2A_2$, if $K_1 \neq A_1$) + K_1A_2 (on $A_1X_1A_2$). Then, as $A_1X_{11}A_2$ fails to meet $C_1Z_1C_2$ and I_1 and I_2 are either both within

* If $m = n$ and $L_{2n-1} \neq L_{2n}$, then L_{2n} is on $L_{2n-1}C_2$ of C_1ZC_2.

or both without $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$, it follows by Theorem E, that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_{11}A_1$. Let P_1 denote a point between K_1 and L_1 on $A_1X_2A_2$.

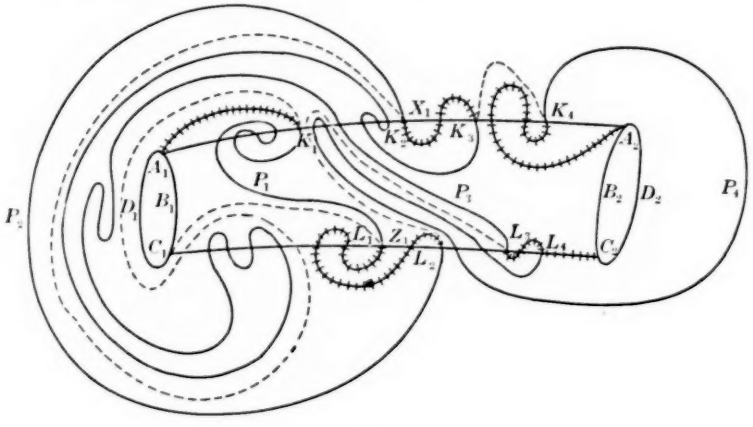


FIG. 1.

If A_2 is without $A_1B_1C_1L_1P_1K_1A_1$, then all points of $\overset{\frown}{K_1A_2}$ (on $A_1X_{11}A_2$) $+ J_2 + \overset{\frown}{C_2L_1}$ (on $C_1Z_1C_2$) are without $A_1B_1C_1L_1P_1K_1A_1$. As the arc L_1A_2 (on $A_1X_2A_2$) fails to have any point in common with $C_1B_1A_1K_1P_1L_1$, then, by Theorem D, there exists an arc $C_1M_1L_1$ such that $\overset{\frown}{C_1M_1L_1}$ is within $A_1B_1C_1L_1P_1K_1A_1$ and fails to meet either $\overset{\frown}{L_1A_2}$ (on $A_1X_2A_2$) or $\overset{\frown}{A_1D_1C_1}$. If A_2 is within $A_1B_1C_1L_1P_1K_1A_1$, then all points of $\overset{\frown}{K_1A_2}$ (on $A_1X_{11}A_2$) $+ J_2 + \overset{\frown}{C_2L_1}$ (on $C_1Z_1C_2$) are within $A_1B_1C_1L_1P_1K_1A_1$. In this case let $C_1M_1L_1$ denote an arc such that $\overset{\frown}{C_1M_1L_1}$ is without $A_1B_1C_1L_1P_1K_1A_1$ and fails to meet either $\overset{\frown}{L_1A_2}$ or $\overset{\frown}{A_1D_1C_1}$. Let $C_1Z_{11}C_2$ denote the arc composed of $C_1M_1L_1 + \overset{\frown}{L_1C_2}$ (on $C_1Z_1C_2$). As $C_1Z_{11}C_2$ fails to meet $A_1X_{11}A_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1C_2B_2A_2X_{11}A_1$, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{11}C_2B_2A_2X_{11}A_1$. Let K_2 denote the first point of the arc L_1A_2 of $A_1X_2A_2$ which is on $A_1X_{11}A_2$. Let L_2 denote the last point of the arc $C_1Z_{11}C_2$ which is on the arc L_1K_2 of $A_1X_2A_2$. Clearly L_2 either is L_1 or is a point of $\overset{\frown}{L_1C_2}$ on $C_1X_{11}C_2$. Let $C_1Z_{12}C_2$ denote the arc composed of C_1L_1 (on $\overset{\frown}{C_1Z_{11}C_2}$) $+ L_1L_2$ (on $A_1X_2A_2$, if $L_1 \neq L_2$) $+ L_2C_2$ (on $C_1Z_{11}C_2$). As $C_1Z_{12}C_2$ fails to meet $A_1X_{11}A_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{11}C_2B_2A_2X_{11}A_1$, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{12}C_2B_2A_2X_{11}A_1$. Let P_2 denote a point between L_2 and K_2 on $A_1X_2A_2$.

If A_2 is without $K_1P_1L_1L_2P_2K_2K_1$, then all points of K_2A_2 (on $A_1X_{11}A_2$) $+ J_2 + C_2L_2$ (on $C_1Z_{12}C_2$) are without $K_1P_1L_1L_2P_2K_2K_1$. Let $K_1F_2K_2$ denote an arc such that $K_1F_2K_2$ is within $K_1P_1L_1L_2P_2K_2K_1$ and fails to meet K_2A_2 (on $A_1X_2A_2$) $+ K_1A_1B_1C_1L_1 + A_1D_1C_1$. If A_2 is within $K_1P_1L_1L_2P_2K_2K_1$, then all points of K_2A_2 (on $A_1X_{11}A_2$) $+ J_2 + C_2L_2$ (on $C_1Z_{12}C_2$) are within $K_1P_1L_1L_2P_2K_2K_1$. In this case let $K_1F_2K_2$ denote an arc such that $K_1F_2K_2$ is without $K_1P_1L_1L_2P_2K_2K_1$, and fails to meet K_2A_2 (on $A_1X_2A_2$) $+ K_1A_1B_1C_1L_1 + A_1D_1C_1$. Let $A_1X_{12}A_2$ denote the arc composed of AK_1 (on $A_1X_{11}A_2$) $+ K_1F_2K_2 + K_2A_2$ (on $A_1X_{11}A_2$). As $A_1X_{12}A_2$ fails to meet $C_1Z_{12}C_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{12}C_2B_2A_2X_{11}A_1$, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{12}C_2B_2A_2X_{12}A_1$.

Continue this process, always being sure that if A_2 is $\{^{\text{without}}_{\text{within}}\}$ $L_{2i}P_{2i}K_{2i}K_{2i+1}P_{2i+1}L_{2i+1}L_{2i}$, then $L_{2i}M_{2i+1}L_{2i+1}$ is an arc such that $L_{2i}M_{2i+1}L_{2i+1}$ is $\{^{\text{within}}_{\text{without}}\}$ $L_{2i}P_{2i}K_{2i}K_{2i+1}P_{2i+1}L_{2i+1}L_{2i}$ and fails to meet $L_{2i+1}A_2$ (on $A_1X_2A_2$) $+ L_{2i}A_1B_1C_1K_{2i} + A_1D_1C_1 + K_1P_1L_1 + K_2P_2L_2 + \cdots + K_{2i-1}P_{2i-1}L_{2i-1}$; and if A_2 is $\{^{\text{without}}_{\text{within}}\}$ $K_{2i+1}P_{2i+1}L_{2i+1}L_{2(i+1)}P_{2(i+1)}K_{2(i+1)}K_{2i+1}$, then $K_{2i+1}F_{2(i+1)}K_{2(i+1)}$ is an arc such that $K_{2i+1}F_{2(i+1)}K_{2(i+1)}$ is $\{^{\text{within}}_{\text{without}}\}$ $K_{2i+1}P_{2i+1}L_{2i+1}L_{2(i+1)}P_{2(i+1)}K_{2(i+1)}K_{2i+1}$ and fails to meet $K_{2(i+1)}A_2$ (on $A_1X_2A_2$) $+ K_{2i+1}A_1B_1C_1L_{2i+1} + A_1D_1C_1 + K_1P_1L_1 + K_2P_2L_2 + \cdots + K_{2i}P_{2i}L_{2i}$.

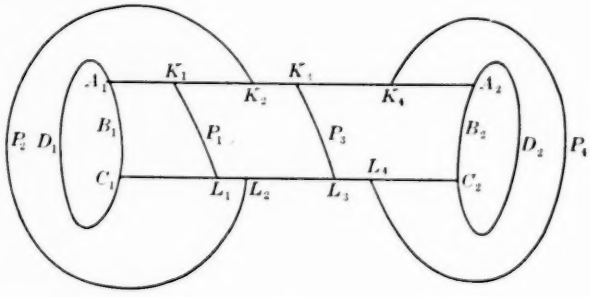


FIG. 2.

By Theorem A, there exists a certain finite number n such that on the arc $K_{2n}A_2$ of $A_1X_2A_2$, there is no point of the arc $C_1Z_1, {}_{2n}C_2$. It is clear the K_{2n} is on the arc $K_{2n-1}A_2$ of $A_1X_1, {}_{2n}A_2$ while I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1, {}_{2n}C_2B_2A_2X_1, {}_{2n}A_2$. Let A_1XA_2 denote the arc composed of A_1K_{2n} (on $A_1X_1, {}_{2n}A_2$) $+ K_{2n}A_2$ (on $A_1X_2A_2$) while C_1ZC_2 denotes $C_1Z_1, {}_{2n}C_2$. As A_1XA_2 fails to meet C_1ZC_2 and I_1 and I_2 are either both within or both without $A_1B_1C_1Z_1, {}_{2n}C_2B_2A_2X_1, {}_{2n}A_1$,

it is clear that I_1 and I_2 are either both within or both without $A_1B_1C_1ZC_2B_2A_2XA_1$. It is clear that the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$ has the properties of the closed curve described in the conclusion of our theorem (Fig. 2).

Theorem G: *Under the same hypothesis as in Theorem F, there exists at least one arc $C_1Z_2C_2$ such that* (1) *$C_1Z_2C_2$ fails to meet $A_1X_2A_2$,* (2) *$C_1Z_2C_2$ is a subset of E_{12},* (3) *I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.*

Proof: Suppose the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$, which was described in Theorem F, has been constructed. We shall first prove that our theorem is true when $n = 1$.

Let us suppose I_1 and I_2 are both without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then $A_1D_1C_1$ and $A_2D_2C_2$ are both without $A_1B_1C_1ZC_2B_2A_2XA_1$ and the interior of $A_1D_1C_1ZC_2D_2A_2XA_1 = A_1B_1C_1 + A_2B_2C_2 + I_1 + I_2 +$ the interior of $A_1B_1C_1ZC_2B_2A_2XA_1$. There are a number of possible cases.

Case I. $K_1P_1L_1$ and $K_2P_2L_2$ are both within $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exist an arc $C_1Z_2C_2$ such that $C_1Z_2C_2$ is without $A_1D_1C_1ZC_2D_2A_2XA_1$. By Theorem E, I_1 and I_2 are both within or both without $A_1B_1C_1Z_2C_2B_2A_2XA_1$. But as $A_1X_2A_2$ fails to meet $C_1Z_2C_2$, it follows, by Theorem E, that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Case II. $K_1P_1L_1$ and $K_2P_2L_2$ are both without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then there exists an arc $C_1Z_2C_2$ such that $C_1Z_2C_2$ is within $A_1B_1C_1ZC_2B_2A_2XA_1$. By Theorem E, I_1 and I_2 are both within or both without $A_1B_1C_1Z_2C_2B_2A_2XA_1$. But as $A_1X_2A_2$ fails to meet $C_1Z_2C_2$, it follows, by Theorem E, that I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Case III. $K_1P_1L_1$ is within and $K_2P_2L_2$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$. Then the interior of $A_1B_1C_1ZC_2B_2A_2XA_1 = K_1P_1L_1 +$ the interior of $A_1B_1C_1L_1P_1K_1A_1 +$ the interior of $A_2B_2C_2L_1P_1K_1A_2$. As $K_2P_2L_2$ is without $A_1B_1C_1ZC_2B_2A_2XA_1$ and fails to contain any point of $A_1D_1C_1 + A_2D_2C_2 + I_1 + I_2$, it follows that $K_2P_2L_2$ is without $A_1D_1C_1ZC_2D_2A_2XA_1$. There are two possibilities:

(*a*) $A_1D_1C_1$ is without $A_2D_2C_2L_2P_2K_2A_2$. Then the interior of $A_1D_1C_1L_2P_2K_2A_1 = K_2A_2D_2C_2L_2 +$ the interior of $A_1D_1C_1ZC_2D_2A_2XA_1 +$ the interior of $A_2D_2C_2L_2P_2K_2A_2$.* Let M denote a point between K_1

* Cf. R. L. Moore, Theorem 27, pp. 144–5.

and K_2 on A_1XA_2. Let C_1M be an arc such that $\overset{\frown}{C_1M}$ is a subset of the exterior of $A_1D_1C_1L_2P_2K_2A_1$ while MC_2 is an arc such that $\overset{\frown}{MC_2}$ is a subset of the interior of $A_2B_2C_2L_1P_1K_1A_2$. Let F denote a point between C_1 and L_1 on C_1ZC_2 while R denotes a point between M and K_2 on A_1XA_2. Let B_1F denote an arc such that $\overset{\frown}{B_1F}$ is a subset of the interior of $A_1B_1C_1$-$L_1P_1K_1A_1$. As $\overset{\frown}{MC_2}$ is within $A_2B_2C_2L_1P_1K_1A_2$, the interior of $A_2B_2C_2$-$L_1P_1K_1A_2 = \overset{\frown}{MC_2}$ + the interior of $MK_2A_2B_2C_2M$ + the interior of $K_1P_1L_1C_2MK_1$. Let RB_2 denote an arc such that $\overset{\frown}{RB_2}$ is a subset of the interior of $MK_2A_2B_2C_2M$. Let FR denote an arc such that $\overset{\frown}{FR}$ is within $C_1L_2P_2K_2MC_1$, provided $\overset{\frown}{A_1D_1C_1}$ is without $C_1L_2P_2K_2MC_1$, while $\overset{\frown}{FR}$ is without $C_1MK_2P_2L_2C_1$ provided $\overset{\frown}{A_1D_1C_1}$ is within $C_1MK_2P_2L_2C_1$. Let B_1YB_2 denote the arc composed of $\overset{\frown}{B_1F} + \overset{\frown}{FR} + \overset{\frown}{RB_2}$. Then B_1YB_2, $A_1X_2A_2$ and C_1MC_2 are arcs no two of which have any point in common and lying except for their endpoints in E_{12}. By Theorem B, I_1 and I_2 are either both within or both without $A_1B_1C_1MC_2B_2A_2X_2A_1$.

(*b*) $\overset{\frown}{A_1D_1C_1}$ is within $A_2D_2C_2L_2P_2K_2A_2$. Then the interior of $A_2D_2C_2$-$L_2P_2K_2A_2 = \overset{\frown}{K_2A_1D_1C_1L_2}$ + the interior of $A_1D_1C_1ZC_2D_2A_2XA_1$ + the interior of $K_2A_1D_1C_1L_2P_2K_2$. Let M denote a point between K_1 and K_2 on A_1XA_2. Let C_1M be an arc such that $\overset{\frown}{C_1M}$ is within $A_1D_1C_1L_2P_2K_2A_1$, whose interior then is $\overset{\frown}{C_1M}$ + the interior of $C_1L_2P_2K_2MC_1$ + the interior of $A_1D_1C_1MA_1$. Let C_2M denote an arc such that $\overset{\frown}{C_2M}$ is within $A_2B_2C_2$-$L_1P_1K_1A_2$, whose interior then is $\overset{\frown}{MC_2}$ + the interior of $A_2B_2C_2MK_2A_2$ + the interior of $K_1MC_2L_1P_1K_1$. Let F denote a point between C_1 and L_1 on C_1ZC_2 while B_1F is an arc such that $\overset{\frown}{B_1F}$ is within $A_1B_1C_1L_1P_1K_1A_1$. Let R denote a point between M and K_2 on A_1XA_2 while RF is an arc such that $\overset{\frown}{RF}$ is within $C_1L_2P_2K_2MC_1$. Let RB be an arc such that $\overset{\frown}{RB_2}$ is within $A_2B_2C_2MK_2A_2$. Let B_1YB_2 denote the arc $B_1F + FR + \overset{\frown}{RB_2}$. Then as $A_1X_2A_2$, B_1YB_2 and C_1MC_2 are arcs lying except for their endpoints in E_{12} and such that no two have any point in common, it follows that I_1 and I_2 are either both within or both without $A_1B_1C_1MC_2B_2A_2XA_1$.

Case IV. $\overset{\frown}{K_1P_1L_1}$ is without and $\overset{\frown}{K_2P_2L_2}$ is within $A_1B_1C_1Z_1C_2B_2A_2$-XA_1. Case IV reduces to Case III if we read from A_2 to A_1 instead of from A_1 to A_2 on $A_1X_2A_2$.

It may readily be seen that in case I_1 and I_2 are both within $A_1B_1C_1$-$ZC_2B_2A_2XA_1$, then I_1 and I_2 are both without $A_1D_1C_1ZC_2D_2A_2XA_1$. The argument just given shows that there exists an arc $C_1Z_2C_2$ such that (1) $\overset{\frown}{C_1Z_2C_2}$ fails to meet $A_1X_2A_2$ and is a subset of E_{12}, (2) I_1 and I_2 are

either both within or both without $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. Hence by Theorems B and C, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Having now proved that our theorem is true when $n = 1$, let us assume that it holds true when $n = k - 1$, to prove it holds true for $n = k$. The arc A_1XA_2 composed of A_1K_2 (on A_1XA_2) + K_2A_2 (on $A_1X_2A_2$) will be an arc for which $n = k - 1$. Hence there exists an arc C_1ZC_2 such that (1) C_1ZC_2 fails to meet A_1XA_2, (2) $C_1\underset{\frown}{ZC_2}$ is a subset of E_{12}, (3) I_1 and I_2 are either both within or both without $A_1B_1C_1\underset{\frown}{ZC_2}B_2A_2XA_1$.

If no point of A_1K_2 (on $A_1X_2A_2$) is on C_1ZC_2, then by Theorem E, I_1 and I_2 are either both within or both without $A_1B_1C_1ZC_2B_2A_2X_2A_1$. Suppose now that there is a point of AK_2 (on $A_1X_2A_2$) on C_1ZC_2. Let S_1 be the first such point. It is evident that K_1 is the last point of the arc A_1XA_2 on the arc A_1S_1 of $A_1X_2A_2$, while K_2 will be the first point of the arc S_1A_2 of $A_1X_2A_2$ on A_1XA_2. Proceeding now with S_1 as we did with L_1 in Theorem F, it is evident that we will get a closed curve $A_1B_1C_1Z'C_2B_2A_2X'A_1$ having all the properties of the closed curve $A_1B_1C_1ZC_2B_2A_2XA_1$ of Theorem F and having the additional property that $n = 1$. Hence by the proof carried on in the first part of this theorem, there exists an arc $C_1Z_2C_2$ failing to meet $A_1X_2A_2$ and such that (1) $C_1\underset{\frown}{Z_2C_2}$ is a subset of E_{12}, (2) I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$. Thus, if we assume our theorem is true when $n = k - 1$, it is also true when $n = k$. But it is true when $n = 1$. Hence our induction is complete.

Theorem H: *Suppose* (1) A_i, B_i *and* C_i *are distinct points of the closed curve* J_i $(i = 1, 2)$, (2) $J_1 + I_1$ *is a subset of* E_2, (3) $A_1X_1A_2$, $A_1X_2A_2$, $C_1Z_1C_2$ *and* $C_1Z_2C_2$ *are arcs which lie except for their endpoints in* E_{12}, (4) $A_1X_iA_2$ *and* $C_1Z_iC_2$ $(i = 1, 2)$ *have no points in common,* (5) I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_1C_2B_2A_2X_1A_1$. *Under these conditions,* I_1 *and* I_2 *are either both within or both without* $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Proof: By Theorem G there exists at least one arc C_1EC_2 such that (1) C_1EC_2 fails to meet $A_1X_2A_2$, (2) $C_1\underset{\frown}{EC_2}$ is a subset of E_{12}, (3) I_1 and I_2 are either both within or both without $A_1B_1C_1EC_2B_2A_2X_2A_1$. Hence, by Theorem E, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$.

Definition: If A_i, B_i and C_i are distinct points of J_i $(i = 1, 2)$, where J_1 and J_2 are closed curves such that $J_1 + I_1$ is a subset of E_2, then we shall say that $A_1B_1C_1$ can be *simply joined* to $A_2B_2C_2$ if there exist arcs

A_1XA_2, B_1YB_2 and C_1ZC_2, which are such that no two have any point in common and, which lie, except for their endpoints, in E_{12}.

Theorem I: *Suppose* (1) J_1, J_2 *and* J_3 *are closed curves such that* $J_i + I_i$ *is a subset of* $E_{i+1,\,i+2}$,* (2) A_i, B_i *and* C_i $(i = 1, 2, 3)$ *are distinct points of* J_i, (3) $A_1B_1C_1$ *can be simply joined to* $A_3B_3C_3$ *and* $A_2B_2C_2$ *can be simply joined to* $A_3B_3C_3$. *Then* $A_1B_1C_1$ *and* $A_2B_2C_2$ *cannot be simply joined.*

Proof: Draw arcs $A_1X_{13}A_3$ and $C_1Z_{13}C_3$, having no points in common and such that $A_1X_{13}A_3 + C_1Z_{13}C_3$ is a subset of E_{123}, the common exterior of J_1, J_2 and J_3 and such that J_2 is without the closed curve $A_1B_1C_1$-$Z_{13}C_3B_3A_3X_{13}A_1$.† By Theorems B and H, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Let D_i denote a point of J_i such that A_i and C_i separate B_i and D_i on J_i. There are two cases possible.

Case I. The sets I_1 and I_3 are both without $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Then it may easily be proved that the interior of $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$ $= A_1B_1C_1 + A_3B_3C_3 + I_1 + I_3 +$ the interior of $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Draw arcs $A_2X_{23}A_3$ and $C_2Z_{23}C_3$, having no points in common and such that $A_2X_{23}A_3 + C_2Z_{23}C_3$ is a subset of the common exterior of J_2 and $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. There are two possibilities.

(*a*) The point D_1 is without $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. Then the interior of $A_1D_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$ is $A_3D_3C_3 +$ the interior of $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1 +$ the interior of $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. As $A_3B_3C_3$ is without the closed curve $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$ while B_2 is not within $A_3B_3C_3D_3A_3$, it follows that I_3 is within $A_3B_3C_3Z_{23}C_2B_2A_2$-$X_{23}A_3$.‡ Hence, by Theorems B and H, I_2 must be within $A_3B_3C_3$-$Z_{23}C_2B_2A_2X_{23}A_3$. As $A_1B_1C_1$ is within $A_1D_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3$-$X_{13}A_1$, then I_1 is without $A_1B_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, which we know does contain I_2. Hence, by Theorems B and H, we may conclude that $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.

(*b*) The point D_1 is within $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$, whose interior then is $A_3X_{13}A_1D_1C_1Z_{13}C_3 +$ the interior of $A_2B_2C_2Z_{23}C_3Z_{13}C_1D_1A_1$-$X_{13}A_3X_{23}A_2 +$ the interior of $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. As $A_3B_3C_3$ is within $A_3D_3C_3Z_{23}C_2B_2A_2X_{23}A_3$, I_3 is without $A_3B_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. Hence, by Theorems B and H, I_2 is also without $A_3B_3C_3Z_{23}C_2B_2A_2X_{23}A_3$. As $A_1B_1C_1$ is without $A_1D_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, while B_2 is

* It is understood throughout this and the following theorem, that subscripts are reduced modulo 3.

† That this drawing of arcs is possible, may easily be proved on the basis of Σ_3.

‡ Cf. R. L. Moore, loc. cit., Theorem 27, pp. 144–5.

without $A_1B_1C_1D_1A_1$, I_1 is within $A_1B_1C_1Z_{13}C_3Z_{23}C_2B_2A_2X_{23}A_3X_{13}A_1$, which we know fails to contain I_2. Hence, by Theorems B and H, $A_1B_1C_1$ and $A_2B_2C_2$ are not able to be simply joined.

Case II. I_1 and I_3 are both within $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. Then I_1 and I_3 are both without $A_1D_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. As $A_1B_1C_1$ and $A_3B_3C_3$ are able to be simply joined, $A_1D_1C_1$ and $A_3D_3C_3$ can also be simply joined, by Theorem C. In like manner, $A_2D_2C_2$ and $A_3D_3C_3$ can be simply joined. By Case I, $A_1D_1C_1$ and $A_2D_2C_2$ cannot be simply joined. Hence, by Theorem C, $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.

Theorem J: *Suppose J_1, J_2 and J_3 are closed curves such that $J_i + I_i$ ($i = 1, 2, 3$) is a subset of $E_{i+1,\ i+2}$, (2) A_i and C_i separate B_i and D_i on J_i, (3) $A_3B_3C_3$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$. Then $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.*

Proof: Join A_1 to A_3 and C_1 to C_3 by arcs, $A_1X_{13}A_3$ and $C_1Z_{13}C_3$ which have no point in common and lie except for their endpoints in E_{13}. As $A_1B_1C_1$ and $A_3B_3C_3$ cannot be simply joined, it follows, by Theorems B and H, that either I_1 is within and I_3 is without or I_1 is without and I_3 is within $A_1B_1C_1Z_{13}C_3B_3A_3X_{13}A_1$. It follows that I_1 and I_3 are either both without or both within $A_1B_1C_1Z_{13}C_3D_3A_3X_{13}A_1$. Hence $A_1B_1C_1$ and $A_3D_3C_3$ can be simply joined. In like manner $A_2B_2C_2$ and $A_3D_3C_3$ can be simply joined. It follows, by Theorem I, that $A_1B_1C_1$ and $A_2B_2C_2$ cannot be simply joined.

Theorem K: *Suppose J_1 and J_2 are closed curves such that $J_1 + I_1$ is a subset of E_2, (2) A_i and C_i separate B_i and D_i on J_i, (3) $A_1B_1C_1$ and $A_2B_2C_2$ can be simply joined. Then $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply joined. Conversely if $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply joined, then $A_1B_1C_1$ and $A_2B_2C_2$ can be simply joined.*

Proof: Join A_1 to A_2 and C_1 to C_2 by arcs, $A_1X_2A_2$ and $C_1Z_2C_2$, which have no point in common and lie except for their endpoints in E_{12}. As $A_1B_1C_1$ and $A_2B_2C_2$ can be simply joined, I_1 and I_2 are either both within or both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$. Let us suppose they are both without. Then it may be proved that the interior of $A_1D_1C_1Z_2C_2D_2A_2X_2A_1 = \underbrace{A_1B_1C_1} + \underbrace{A_2B_2C_2} + I_1 + I_2 +$ the interior of $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$. Let $\underbrace{A_1M_1C_2}$ denote an arc such that $\underbrace{A_1M_1C_2}$ is within $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$ while $C_1K_1A_2$ is an arc such that $\underbrace{C_1K_1A_2}$ is without $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. There are two possibilities.

(*a*) X_2 is within $A_2D_2C_2Z_2C_1K_1A_2$, whose interior then is $\underbrace{A_2X_2A_1D_1C_1}$ + interior of $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$ + the interior of $A_2X_2A_1D_1C_1K_1A_2$. As $\underbrace{C_1B_1A_1M_1C_2}$ is within $A_2D_2C_2Z_2C_1K_1A_2$, the interior of $A_2D_2C_2Z_2C_1K_1A_2 = \underbrace{C_1B_1A_1M_1C_2}$ + interior of $C_1B_1A_1M_1C_2Z_2C_1$ + interior of A_1-

$B_1C_1K_1A_2D_2C_2M_1A_1$. As the interior of $C_1B_1A_1M_1C_2Z_2C_1$ is a subset of the interior of $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$,* it follows that I_1 and I_2 together with $A_2B_2C_2$ are within $A_1B_1C_1K_1A_2D_2C_2M_1A_1$. As $A_2B_2C_2$ is within $A_1B_1C_1K_1A_2D_2C_2M_1A_1$, it follows that I_2, the interior of $A_2B_2C_2D_2A_2$ is without $A_1B_1C_1K_1A_2B_2C_2M_1A_1$.† Hence I_1 is within and I_2 is without $A_1B_1C_1K_1A_2B_2C_2M_1A_1$.

(*b*) The point X_2 is without $A_2D_2C_2Z_2C_1K_1A_2$. Then the interior of $A_1D_1C_1K_1A_2X_2A_1 = A_2D_2C_2Z_2C_1$ + the interior of $A_2D_2C_2Z_2C_1K_1A_2$ + the interior of $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. As $A_1M_1C_2B_2A_2$ is within $A_1D_1C_1K_1A_2X_2A_1$, the interior of $A_1D_1C_1K_1A_2X_2A_1 = A_1M_1C_2B_2A_2$ + the interior of $A_1M_1C_2B_2A_2X_2A_1$ + the interior of $A_2B_2C_2M_1A_1D_1C_1K_1A_2$. As I_1, I_2 and $A_1B_1C_1$ are in $A_1D_1C_1K_1A_2X_2A_1$ and not in $A_1M_1C_2B_2A_2X_2A_1$, which lies entirely within $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$, it follows that I_1, I_2 and $A_1B_1C_1$ are within $A_2B_2C_2M_1A_1D_1C_1K_1A_2$. As $A_1B_1C_1$ is within $A_2B_2C_2M_1A_1D_1C_1K_1A_2$, it follows that I_1, the interior of $A_1B_1C_1D_1A_1$ is without $A_2B_2C_2M_1A_1B_1C_1K_1A_2$, which does contain I_2.

Thus, if I_1 and I_2 are both without $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$, it follows that $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply joined. If I_1 and I_2 are both within $A_1B_1C_1Z_2C_2B_2A_2X_2A_1$, I_1 and I_2 are both without $A_1D_1C_1Z_2C_2D_2A_2X_2A_1$. Hence $A_1D_1C_1$ and $C_2D_2A_2$ cannot be simply joined. Hence it follows, by Theorem *C*, that $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply joined.

A similar argument will show that if $A_1B_1C_1$ and $C_2B_2A_2$ cannot be simply connected, then $A_1B_1C_1$ and $A_2B_2C_2$ can be simply connected.

3. Properties of Sense.

Theorem 1: *If for one choice of the closed curve J_3 in E_{12} and three distinct points $A_3B_3C_3$ thereon, $A_1B_1C_1$ on J_1 and $A_2B_2C_2$ on J_2 are either both able or both not able to be simply joined to $A_3B_3C_3$, then for any other closed J_4 in E_{12} and three distinct points $A_4B_4C_4$ thereon $A_1B_1C_1$ and $A_2B_2C_2$ are either both able or both not able to be simply joined to $A_4B_4C_4$.*

Proof: Case I. Suppose $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined. Then $A_2B_2C_2$ and $A_3B_3C_3$ can be simply joined. Choose in the common exterior of J_1, J_2, J_3 and J_4 a closed curve J_5 and three points A_5, B_5 and C_5 thereon such that $A_3B_3C_3$ and $A_5B_5C_5$ can be simply joined. By Theorem *I*, $A_5B_5C_5$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$. There are two possibilities

(*a*) $A_4B_4C_4$ and $A_5B_5C_5$ can be simply joined. Then, by Theorem *K*,

* Cf. R. L. Moore, loc. cit., Theorem 21, p. 140.

† Cf. R. L. Moore, loc. cit., Theorem 24, p. 141.

$C_4B_4A_4$ and $A_5B_5C_5$ cannot be simply joined. By Theorem J, $C_4B_4A_4$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$. By Theorem K, $A_4B_4C_4$ can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

(*b*) $A_4B_4C_4$ and $A_5B_5C_5$ cannot be simply joined. Then, by Theorem J, $A_4B_4C_4$ cannot be simply joined to either $A_1B_1C_1$ or $A_2B_2C_2$.

Case II. Suppose $A_1B_1C_1$ and $A_3B_3C_3$ cannot be simply joined. Then $A_2B_2C_2$ and $A_3B_3C_3$ cannot be simply joined. Then, by Theorem K, $A_3B_3C_3$ can be simply joined to both $C_1B_1A_1$ and $C_2B_2A_2$. Then by an argument, similar to that used in Case I, we may show that either (*a*) $A_4B_4C_4$ can be simply joined to both $C_1B_1A_1$ and $C_2B_2A_2$ or (*b*) $A_4B_4C_4$ cannot be simply joined to either $C_1B_1A_1$ or $C_2B_2A_2$. By Theorem K, either $A_4B_4C_4$ (*a*) can be simply joined to neither $A_1B_1C_1$ nor $A_2B_2C_2$ or (*b*) can be simply joined to both $A_1B_1C_1$ and $A_2B_2C_2$.

Theorem 2: *If the senses $A_1B_1C_1$ on J_1 and $A_2B_2C_2$ on J_2 are the same with respect to their common exterior E_{12} and the senses $A_2B_2C_2$ on J_2 and $A_3B_3C_3$ on J_3 are the same with respect to their common exterior E_{23}, then the senses $A_1B_1C_1$ on J_1 and $A_3B_3C_3$ on J_3 are the same with respect to their common exterior E_{13}.*

Proof: Choose a closed curve J_4 in the common exterior of J_1, J_2 and J_3 and three distinct points A_4, B_4 and C_4 on J_4 such that $A_1B_1C_1$ and $A_4B_4C_4$ can be simply joined. As the sense $A_1B_1C_1$ on J_1 and the sense $A_2B_2C_2$ on J_2 are the same with respect to E_{12}, it follows, by definition and Theorem 1, that $A_2B_2C_2$ and $A_4B_4C_4$ can be simply joined. As the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3 with respect to E_{23}, it follows, by definition and Theorem 1, that $A_3B_3C_3$ and $A_4B_4C_4$ can be simply joined. Hence, by definition, the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_3B_3C_3$ on J_3 with respect to E_{13}.

Theorem 3: *If the sense $A_1B_1C_1$ on J_1 is opposite to the sense $A_2B_2C_2$ on J_2 with respect to E_{12}, and the sense $A_2B_2C_2$ on J_2 is opposite to the sense $A_3B_3C_3$ on J_3 with respect to E_{23}, then the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_3B_3C_3$ with respect to E_{13}.*

Proof: Choose in the common exterior of J_1, J_2 and J_3 a closed curve J_4 and three distinct points A_4, B_4 and C_4 on J_4 such that $A_4B_4C_4$ and $A_1B_1C_1$ can be simply joined. As the sense $A_1B_1C_1$ on J_1 is opposite to the sense $A_2B_2C_2$ on J_2 with respect to E_{12}, it follows that $A_2B_2C_2$ and $A_4B_4C_4$ cannot be simply joined. As the senses $A_2B_2C_2$ on J_2 and $A_3B_3C_3$ on J_3 are opposite with respect to E_{23}, it follows that $A_3B_3C_3$ and $A_4B_4C_4$ can be simply joined. As $A_4B_4C_4$ can be simply joined to $A_3B_3C_3$ and to $A_1B_1C_1$, it follows that the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_3B_3C_3$ on J_3 with respect to E_{13}.

Theorem 4: *The sense $A_1B_1C_1$ on J_1 is opposite to the sense $C_1B_1A_1$ on J_1 with respect to E_1.*

The truth of Theorem 4 follows at once from our definition and Theorems 1 and *K*.

Theorem 5: *If the closed curve J_1 is a subset of the interior of the closed curve J_2, then, in order that the sense $A_1B_1C_1$ on J_1 and the sense $A_2B_2C_2$ on J_2 be the same with respect to E_{12}, it is both necessary and sufficient that there exist arcs A_1XA_2, B_1YB_2 and C_1ZC_2, no two having any points in common and lying except for their endpoints entirely between* J_1 and J_2.*

Proof: *The condition is sufficient.* Let J_3 denote any closed curve in E_2, the exterior of J_2. Choose on J_3 points A_3, B_3 and C_3 such that $A_3B_3C_3$ and $A_2B_2C_2$ can be simply joined. Hence there exist arcs $A_2X_2A_3$, $B_2Y_2B_3$ and $C_2Z_2C_3$ such that (1) these arcs lie except for their endpoints in E_{23}, (2) no two have any points in common, (3) I_2 and I_3 are both without $A_2B_2C_2Z_2C_3B_3A_3X_2A_2$. By hypothesis, there exist arcs B_1YB_2, A_1XA_2 and C_1ZC_2, no two of which have any point in common and lie except for their endpoints within J_2 but without J_1. Hence $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined. Hence the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12}.

The condition is necessary. Suppose the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12}. Consider the closed curve J_3 and the arcs $A_2X_2A_3$, $B_2Y_2B_3$ and $C_2Z_2C_3$ described above. Draw arcs $A_1X_1A_2$ and $C_1Z_1C_2$ which have no point in common and lie except for their endpoints in between J_1 and J_2. As the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2, it follows that, as $A_2B_2C_2$ and $A_3B_3C_3$ can be simply joined, $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined. As $A_2X_1A_1B_1C_1Z_1C_1$ is without $A_3B_3C_3Z_2C_2B_2A_2X_2A_3$, and B_3 is not within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$, then $A_2B_2C_2$ is within $A_1B_1C_1Z_1C_2Z_2C_3B_3A_3X_2A_2X_1A_1$, whose interior is $A_2B_2C_2$ + the interior of $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$ + the interior of $A_2B_2C_2Z_2C_3B_3A_3X_2A_2$. As $A_1B_1C_1$ and $A_3B_3C_3$ can be simply joined while I_3 is without $A_1B_1C_1Z_1C_2Z_2C_3B_3A_3X_2A_2X_1A_1$, it follows by Theorems *B* and *H*, that I_1 is without $A_1B_1C_1Z_1C_2Z_2C_3B_3A_3X_2A_2X_1A_1$ and hence without $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$. But all points within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$ are within J_2. Hence all points within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$ are between J_1 and J_2. But there exists an arc $B_1Y_1B_2$ such that $B_1Y_1B_2$ is within $A_2B_2C_2Z_1C_1B_1A_1X_1A_2$. Hence, if the sense $A_1B_1C_1$ on J_1 is the same as the sense $A_2B_2C_2$ on J_2 with respect to E_{12} and J_1 is in the interior of J_2, there exist arcs $A_1X_1A_2$, $B_1Y_1B_2$ and $C_1Z_1C_2$, no two of which have any point in common and which lie except for their endpoints between J_1 and J_2.

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* A point set M is said to lie *between* two closed curves if one of the curves lies within the other and M lies without the first and within the second one.

COVARIANT EXPANSION OF A MODULAR FORM.

By Oliver E. Glenn.

A complete system of covariants of the total group G of linear transformations with integral coefficients modulo p, a prime number, is composed of*

$$L = x_1^p x_2 - x_1 x_2^p,$$

$$Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2}) \div L = x_1^{p(p-1)} + x_1^{(p-1)(p-1)} x_2^{p-1} + \cdots + x_2^{p(p-1)}.$$

Consider a binary form, of order m, whose coefficients are independent variables:

$$f_m = (a_0, a_1, \cdots, a_m)(x_1, x_2)^m = a_0 x_1^m + a_1 x_1^{m-1} x_2 + \cdots.$$

We propose to treat the problem of determining modular covariants ϕ_1, ϕ_2 of f_m such that the following congruence will hold identically in the a's and in the x's:

$$f_m \equiv Q\phi_1 + L\phi_2 \pmod{p}. \tag{1}$$

Regarding the forms ϕ_1, ϕ_2 in a relation like (1) to be general forms with undetermined coefficients, of respective orders $m - p^2 + p$, $m - p - 1$, it is evident that when $m = p^2$ the identity (1) implies $m + 1$ linear non-homogeneous equations between these coefficients. These linear equations are consistent, and hence ϕ_1, ϕ_2 are uniquely determined. In fact, in order to prove the consistency of this system of equations we have only to construct their matrix M, the elements of M being all 0, 1 or -1. These elements are arranged in M by a simple law such that it is immediately evident that elementary transformations will reduce all elements to zero, excepting those in the principal diagonal, and that all in the diagonal will be $\equiv 1 \pmod{p}$. Hence if D is the determinant of M, $D \not\equiv 0 \pmod{p}$.

When, as in the case $m = p^2$ just mentioned, the quantics φ_1, φ_2 are uniquely determined, they are readily proved† to be formal covariants modulo p of f_m. Also, for a series of particular orders m, such that $m > p^2$, I have determined, non-uniquely, covariant pairs φ_1, φ_2 satis-

* Dickson, Transactions Amer. Math. Society, vol. 12 (1911), p. 75; and Madison Colloquium Lectures, 1913, p. 33.

† Cf. O. E. Glenn, Transactions Amer. Math. Society, vol. 18 (1917), p. 460.

fying (1). We shall give in tables computed covariants φ_1, φ_2 leading to an identity (1), for various cases in which $m \geq p^2$.

Whenever we have covariants* φ_1 and φ_2 of f_m for which (1) is an identity, we shall call (1) a covariant expansion, and φ_1 a principal covariant of f_m; its seminvariant leader being a_0. In general the covariant φ_1 leading to an expansion (1) is not unique. But any two principal covariants are linearly dependent modulo L; indeed they all possess their real roots in common.† For, if φ_1, φ_1' are two such covariants their difference $\varphi_1' - \varphi_1$, being a covariant divisible by x_2, contains the factor L, since the real points on the modular line are conjugate under G. Thus in the expansion corresponding to φ_1', viz.,

$$f_m \equiv Q\varphi_1' + L\varphi_2' \pmod{p}, \tag{2}$$

we may substitute

$$\varphi_1' \equiv \varphi_1 + L\psi \pmod{p}, \tag{3}$$

where ψ is a formal covariant mod p of order $m - p^2 - 1$; and so (2) becomes

$$f_m \equiv Q\varphi_1 + L(Q\psi + \varphi_2') \pmod{p},$$

and the principal covariant in this expansion is φ_1. The latter expansion is identical with (1); for, φ_2 may always be determined by dividing $f_m - Q\varphi_1$ by L, and the quotient modulo p is unique. Hence the covariant expansions are all transformable into a fixed one by congruences of type (3).

In case m is sufficiently large, φ_1 and φ_2 will be of order $\geq p^2$, and then, under restrictions similar to those described above for covariant expansions of f_m, φ_1, φ_2 may themselves be developed in covariant expansions; also, if only one of the covariants φ_1, φ_2 is of order $\geq p^2$, this one may be developed according to any expansion (1) known to exist for its order. Thus we arrive at a covariant expansion of f_m more explicit than (1), viz.,

$$f_m \equiv Q^\alpha\varphi_1 + Q^{\beta-1}L\chi + \cdots + QL^{\gamma-1}\psi + L^\delta\omega \pmod{p}, \tag{4}$$

in which the coefficient forms φ_1, χ, $\cdots$, ψ, ω are covariants of f_m of orders $< p^2$; the principal covariant being φ_1 led by a_0. This expansion is not unique.

The existence and explicit form of the principal formal covariants modulo 2 for the general order m were demonstrated by the present writer

* The covariancy of φ_1 evidently implies that of φ_2.

† Since the roots of $L \equiv 0 \pmod{p}$ are the real residues mod p, and those of $Q \equiv 0$ are the Galoisian imaginaries which are roots of irreducible quadratic congruences mod p, the real roots of $f_m \equiv 0 \pmod{p}$ are also roots of $\varphi_1 \equiv 0$, and such imaginary roots of $f_m \equiv 0$ are also roots of $\varphi_2 \equiv 0 \pmod{p}$ (cf. Dickson, Madison Colloquium Lectures, p. 37).

in the Transactions of the American Mathematical Society, volume 17 (p. 545). For m even, φ_1 is the covariant

$$K_2 = a_0x_1^2 + (a_1 + \cdots + a_{m-1})x_1x_2 + a_mx_2^2,$$

while for m odd it was proved that φ_1 is a cubic covariant

$$K_{m3} = a_0x_1^3 + I_1x_1^2x_2 + I_2x_1x_2^2 + a_mx_2^3,$$

where I_1, I_2 are of somewhat complex general structure but such that $I_1 + I_2$ is congruent to the invariant $a_1 + \cdots + a_{m-1}$. Illustrations of K_2 and K_{m3} are contained in the lists below, showing expansions modulo 2 for forms of the first eleven orders, and of the forms of orders 9, 10, 11 when the modulus is 3. The existence of the principal covariants in the latter three cases was established by methods similar to those employed for the modulus 2, but for $m = 11$, in order to exhibit a complete set of principal covariants in one formula, by retaining a parameter λ in their coefficients, φ_1 was assumed in its general form and the conditions for its invariancy under the induced group were imposed. This method required the solution of a set of linear congruences in twenty unknowns, and was not brief, but, once these covariants are found, the direct verification of their covariancy is easy.

Tables.

We employ the notation $(hijk\cdots)$ for the sum

$$a_h + a_i + a_j + a_k + \cdots.$$

$p = 2, \qquad m = 1, 2, 3.$

The forms f_1, f_2, f_3 are irreducible.

$p = 2, \qquad m = 2^2.$

$$f_4 \equiv QK_2 + L(K_1 + C_{101}) \pmod 2.$$

$$K_2 \equiv (0)x_1^2 + (123)x_1x_2 + (4)x_2^2,$$

$$K_1 \equiv (0123)x_1 + (1234)x_2,$$

$$C_{101} \equiv (1)x_1 + (3)x_2.$$

$p = 2, \qquad m = 5.$

$$f_5 \equiv QK_{53} + L(K_2 + C_{102}) \pmod 2.$$

$$K_{53} \equiv (0)x_1^3 + (12)x_1^2x_2 + (34)x_1x_2^2 + (5)x_2^3,$$

$$K_2 \equiv (0)x_1^2 + (1234)x_1x_2 + (5)x_2^2,$$

$$C_{102} \equiv (2)x_1^2 + (3)x_2^2.$$

$$p = 2, \qquad m = 6.$$

$$f_6 \equiv Q^2K_2 + LK_3 \pmod 2.$$

$$K_2 \equiv (0)x_1^2 + (12345)x_1x_2 + (6)x_2^2,$$

$$K_3 \equiv (2345)x_1^3 + (03456)x_1^2x_2 + (01236)x_1x_2^2 + (1234)x_2^3.$$

$$p = 2, \qquad m = 7.$$

$$f_7 \equiv Q^2K_{73} + LQM + L^2J \pmod 2.$$

$$K_{73} \equiv (0)x_1^3 + (124)x_1^2x_2 + (356)x_1x_2^2 + (7)x_2^3,$$

$$M \equiv (24)x_1^2 + (35)x_2^2,$$

$$J \equiv (02356)x_1 + (12457)x_2.$$

$$p = 2, \qquad m = 8.$$

$$f_8 \equiv Q^3K_2 + QLK_{53}' + L^2(K_2' + C_{102}') \pmod 2.$$

$$K_2 \equiv (0)x_1^2 + (1234567)x_1x_2 + (8)x_2^2,$$

$$K_{53}' \equiv (0234567)x_1^3 + (038)x_1^2x_2 + (058)x_1x_2^2 + (1234568)x_2^3,$$

$$K_2' \equiv (0234567)x_1^2 + (35)x_1x_2 + (1234568)x_2^2,$$

$$C_{102}' \equiv (123)x_1^2 + (567)x_2^2.$$

$$p = 2, \qquad m = 9.$$

$$f_9 \equiv Q^3K_{93} + Q^2LK_2' + L^2K_3' \pmod 2.$$

$$K_{93} \equiv (0)x_1^3 + (1234)x_1^2x_2 + (5678)x_1x_2^2 + (9)x_2^3,$$

$$K_2' \equiv (0234)x_1^2 + (124578)x_1x_2 + (5679)x_2^2,$$

$$K_3' \equiv (046)x_1^3 + (157)x_1^2x_2 + (248)x_1x_2^2 + (359)x_2^3.$$

$$p = 2, \qquad m = 10.$$

$$f_{10} \equiv Q^4K_2 + Q^2LK_{73}' + QL^2M' + L^3J' \pmod 2.$$

$$K_2 \equiv (0)x_1^2 + (123456789)x_1x_2 + (10)x_2^2,$$

$$K_{73}' \equiv (23456789)x_1^3 + (0124510)x_1^2x_2 + (0568910)x_1x_2^2 + (12345678)x_2^3,$$

$$M' \equiv (01236789)x_1^2 + (123478910)x_2^2,$$

$$J' \equiv (0235689)x_1 + (12457810)x_2.$$

$$p = 2, \qquad m = 11.$$

$$f_{11} \equiv Q^4K_{113} + Q^3LK_2' + QL^2K_{53}'' + L^3(K_2'' + C_{102}'')(\text{mod } 2).$$

$$K_{113} \equiv (0)x_1^3 + (12458)x_1^2x_2 + (367910)x_1x_2^2 + (11)x_2^3,$$

$$K_2' \equiv (2458)x_1^2 + (56)x_1x_2 + (3679)x_2^2,$$

$$K_{53}'' \equiv (2357910)x_1^3 + (02341011)x_1^2x_2 + (0178911)x_1x_2^2 + (124689)x_2^3,$$

$$K_2'' \equiv (2357910)x_1^2 + (123478910)x_1x_2 + (124689)x_2^2,$$

$$C_{102}'' \equiv (035678910)x_1^2 + (123456811)x_2^2.$$

$$p = 3, \qquad m = 3^2.$$

$$f_9 \equiv Q\varphi_1 + L\varphi_2 \text{ (mod 3)}.$$

$$\varphi_1 \equiv (0)x_1^3 + (1357)x_1^2x_2 + (2468)x_1x_2^2 + (9)x_2^3,$$

$$\varphi_2 \equiv 2(a_3 + a_5 + a_7)x_1^5 + 2(a_0 + a_4 + a_6 + a_8)x_1^4x_2$$
$$+ (2a_1 + 2a_3 + a_5 + a_7 + 2a_9)x_1^3x_2^2$$
$$+ (a_0 + 2a_2 + 2a_4 + a_6 + a_8)x_1^2x_2^3$$
$$+ (a_1 + a_3 + a_5 + a_9)x_1x_2^4 + (a_2 + a_4 + a_6)x_2^5.$$

$$p = 3, \qquad m = 10.$$

$$f_{10} \equiv Q\varphi_1 + L\varphi_2 \text{ (mod 3)}.$$

$$\varphi_1 \equiv a_0x_1^4 + (a_1 + a_3 + 2a_5)x_1^3x_2 + (a_2 + a_4 + a_6 + a_8)x_1^2x_2^2$$
$$+ (2a_5 + a_7 + a_9)x_1x_2^3 + a_{10}x_2^4,$$

$$\varphi_2 \equiv (2a_3 + a_5)x_1^6 + 2(a_0 + a_4 + a_6 + a_8)x_1^5x_2$$
$$+ 2(a_1 + a_3 + a_7 + a_9)x_1^4x_2^2$$
$$+ (a_0 + 2a_2 + 2a_4 + a_6 + a_8 + 2a_{10})x_1^3x_2^3$$
$$+ (a_1 + a_3 + a_7 + a_9)x_1^2x_2^4 + (a_2 + a_4 + a_6 + a_{10})x_1x_2^5$$
$$+ (2a_5 + a_7)x_2^6.$$

$$p = 3, \qquad m = 11.$$

The abbreviations employed are as follows: λ is any least residue modulo 3; S represents the seminvariant $a_1 + a_3 + a_5 + a_7 + a_9$, and T the anti-seminvariant $a_2 + a_4 + a_6 + a_8 + a_{10}$.

$$f_{11} \equiv Q\varphi_1 + L\varphi_2 \text{ (mod 3)}.$$

$$\varphi_1 \equiv a_0x_1^5 + (a_1 + a_3 + \lambda S)x_1^4x_2 + (a_2 + a_4 + a_6 - \lambda T)x_1^3x_2^2$$
$$+ (a_5 + a_7 + a_9 - \lambda S)x_1^2x_2^3 + (a_8 + a_{10} + \lambda T)x_1x_2^4 + a_{11}x_2^5,$$
$$\varphi_2 \equiv (-a_3 - \lambda S)x_1^7 + (-a_0 - a_4 - a_6 + \lambda T)x_1^6x_2 - (\lambda + 1)Sx_1^5x_2^2$$
$$+ (a_0 - a_6 + (\lambda + 2)T)x_1^4x_2^3 + (a_5 - a_{11} - (\lambda + 2)S)x_1^3x_2^4$$
$$+ (\lambda + 1)Tx_1^2x_2^5 + (a_5 + a_7 + a_{11} - \lambda S)x_1x_2^6 + (a_8 + \lambda T)x_2^7.$$

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THE INTERSECTIONS OF A STRAIGHT LINE AND HYPERQUADRIC.*

By G. M. Green.

In a recent note,† Coolidge obtained explicit expressions for the coördinates of the points of intersection of a line and a hyperquadric in space of n dimensions. In the present note we obtain Coolidge's formula by a simple method, suggested by geometric considerations. In fact, our discussion makes clear the geometric significance of all of Coolidge's symbols, including his undetermined multipliers $z_1, z_2, \cdots, z_n$.

Let Q be a hyperquadric, and l a line, in n-space. The line l cuts the quadric in two points M and N whose coördinates we wish to determine. Let α be a point of the line l not on the quadric, and let β be the harmonic conjugate of α with respect to M and N. In homogeneous coördinates the equation of the quadric Q may be written

$$\sum_{i,j} a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}. \tag{1}$$

In this, as in all other summations to follow, the indicated indices are to run independently from 0 to n inclusive. If the coördinates of α and β are $\alpha_0, \alpha_1, \cdots, \alpha_n$ and $\beta_0, \beta_1, \cdots, \beta_n$, the coördinates of any point of the line l will be of the form $\beta_i + p\alpha_i$ $(i = 0, 1, \cdots, n)$. We wish to determine p so that the point shall lie on the quadric. Substituting its coördinates in equation (1), we find that the equation for p is the quadratic

$$p^2 \sum_{i,j} a_{ij}\alpha_i\alpha_j + \sum_{i,j} a_{ij}\beta_i\beta_j = 0, \tag{2}$$

from which the linear term is absent because the points α, β separate harmonically the points M, N. It remains merely to obtain suitable expressions for the quantities α and β in the above formulas.

With Coolidge, let us define the line l by the $n - 1$ linear equations

$$\sum_i u_i^{(1)}x_i = 0, \qquad \sum_i u_i^{(2)}x_i = 0, \qquad \cdots, \qquad \sum_i u_i^{(n-1)}x_i = 0, \tag{3}$$

and let us write

* Presented to the American Mathematical Society, April 28, 1917.

† Cf. these *Annals*, Series 2, vol. 18 (1916–17), pp. 209–13.

$$\Delta = \begin{matrix} v_0, & v_1, & \cdots, & v_n \\ w_0, & w_1, & \cdots, & w_n \\ u_0^{(1)}, & u_1^{(1)}, & \cdots, & u_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_0^{(n-1)}, & u_1^{(n-1)}, & \cdots, & u_n^{(n-1)} \end{matrix}, \tag{4}$$

$$p_{ij} = -p_{ji} = \frac{1}{\Delta}\left(\frac{\partial \Delta}{\partial v_i}\frac{\partial \Delta}{\partial w_j} - \frac{\partial \Delta}{\partial v_j}\frac{\partial \Delta}{\partial w_i}\right). \tag{5}$$

The quantity p_{ij} is obviously, except possibly for sign, the $(n-1)$-rowed determinant formed from the last $n-1$ rows of Δ by deleting the columns with lower indices i and j.

Using expressions of the form (5), which are merely generalized Plücker coördinates of the line l, we may easily establish the identity

$$p_{ij}p_{kl} + p_{ik}p_{lj} + p_{il}p_{jk} = 0, \tag{6}$$

exactly as is done in the case of the identity connecting the Plücker coördinates in ordinary space.

The point α on the line l may now be determined by adjoining to equations (3) another linear equation,

$$\sum_i z_i x_i = 0, \tag{7}$$

which is linearly independent of equations (3). Solving equations (3) and (7) for the ratios of the x's, we obtain for the coördinates α_i of the point α the expressions

$$\alpha_i = \sum_j z_j p_{ij} \qquad (i = 0, 1, \cdots, n). \tag{8}$$

Since the point α is not on the quadric,

$$\sum_{i,j} a_{ij}\alpha_i\alpha_j \neq 0. \tag{9}$$

The polar of α with respect to the quadric is the hyperplane

$$\sum_{i,j} a_{ij}\alpha_i x_j = 0, \tag{10}$$

and the intersection of this with the line l gives the point β, whose coordinates are found by solving equations (3) and (10) for the x's, and are therefore

$$\beta_i = \sum_{j,k} a_{jk}\alpha_k p_{ij} \qquad (i = 0, 1, \cdots, n). \tag{11}$$

In virtue of (8), we may write

$$\beta_i = \sum_{j,k,l} a_{jk} z_l p_{ij} p_{kl} \qquad (i = 0, 1, \cdots, n). \tag{12}$$

Equations (11) and (12) being written in the form

$$\beta_i = \sum_{k,l} a_{kl} \alpha_l p_{ik}, \qquad \beta_j = \sum_{r,s,t} a_{rs} z_t p_{jr} p_{st}, \tag{13}$$

we have

$$\sum a_{ij} \beta_i \beta_j = \sum a_{ij} a_{kl} a_{rs} \alpha_l z_t p_{ik} p_{st} p_{jr}, \tag{14}$$

in which the summations are taken with respect to all the subscripts. By (6), we may put

$$p_{ik} p_{st} = p_{is} p_{kt} - p_{it} p_{ks},$$

so that (14) becomes

$$\sum a_{ij} \beta_i \beta_j = \sum a_{ij} a_{kl} a_{rs} \alpha_l z_t p_{is} p_{kt} p_{jr} - \sum a_{ij} a_{kl} a_{rs} \alpha_l z_t p_{it} p_{ks} p_{jr}. \tag{15}$$

In the right-hand member, the second summation obviously becomes the same as the right-hand member of (14), if the indices i and s are interchanged, and likewise the indices j and r.* Therefore, transposing the second term of the right-hand member of (15) to the other side, and rearranging the first term slightly, we obtain the equation

$$2 \sum a_{ij} \beta_i \beta_j = \sum a_{ij} a_{rs} p_{is} p_{jr} (a_{kl} \alpha_l z_t p_{kt}). \tag{16}$$

The indices k, l, t appear only in the part enclosed in parentheses, therefore that part may be taken out as a factor† $\sum a_{kl} \alpha_l z_t p_{kt}$. But by (8) this factor may be written $\Sigma\, a_{kl} \alpha_k \alpha_l$, so that equation (2) finally becomes, in virtue of (9),

$$p^2 + \tfrac{1}{2} \sum a_{ij} a_{rs} p_{is} p_{jr} = 0.$$

We may therefore state the final result as follows:

The line l intersects the hyperquadric Q in two points whose coördinates are given by

$$x_i = \beta_i + p\alpha_i,$$

where

$$\alpha_i = \sum_j z_j p_{ij}, \qquad \beta_i = \sum_{j,k,l} z_l a_{jk} p_{ij} p_{kl},$$

$$p^2 + \tfrac{1}{2} \sum_{i,j,k,l} a_{ij} a_{kl} p_{il} p_{jk} = 0.$$

HARVARD UNIVERSITY,
April, 1917.

* It should be remembered that by (5) $p_{ij} = -p_{ji}$, and also that $a_{ij} = a_{ji}$.

† This factorization of $\Sigma a_{ij} \beta_i \beta_j$ yields a very general theorem on determinants, only special cases of which the writer is aware of in the literature.

NUMERICAL FUNCTIONS OF $[x]$.

By E. T. Bell.

Theorems on $[x]$, the greatest positive integer contained in x, are broadly of two kinds: (A) Those which reduce sums $\Sigma[f(x)]$; (B) Those which express sums, $\Sigma f(n)$ as functions of $[x]$, where in (A), (B), x assumes successively each of a given class of values, not necessarily integral, and n runs through all members of a given set of integers. (A) has been treated in detail by J. Hacks,* following Dirichlet; and there are numerous instances of (B), due among others to Césaro, Dirichlet, Gegenbauer, Hermite, Liouville, Lipschitz and Mertens.† The derivation of a result in (B) is frequently the first step, and the application of a theorem in (A) to this, the second, in finding arithmetical mean values. Independently of these and other well-known applications, theorems in (B) often have an intrinsic worth—for their elegance, if for no other reason. We shall consider (B) and give simple ways of finding such theorems in any desired number. The principal results of the paper are expressed in equations (1) to (4); (9); (11) to (15); (18), (19); (26), (27), (28). Illustrations of these have been limited to theorems on well-known elementary functions ψ, satisfying $\psi(1) = 1$; $\psi(mn) = \psi(m)\psi(n)$ for m, n relatively prime; we shall call such functions ψ's.

1. **Notation.** $\sum_n$ being over all divisors d, of n, we write:

$$\sum_n g(d)f(n/d) \equiv \overline{gf}(n), \ \equiv \overline{gf},$$

when unnecessary to indicate n; and if $u_r(n) \equiv n^r$, also $u_0 f(n) \equiv \bar{f}(n)$, $= \sum_n f(d)$; and

$$f'(n) \equiv \sum_{a=1}^{n} f(a), \qquad \overline{gf}'(n) \equiv \sum_{a=1}^{n} \overline{gf}(a).$$

Hence obviously, $\overline{gf}'(n)$ denotes $\sum_n g(d)f(\delta)$ over all positive integral d, δ such that $d\delta \equiv n$; also $\overline{gf} = \bar{f}\bar{g}$.

* Acta Math., 10; pp. 1–53.

† References in the Zahltheoretische Funktionen section of the Encyclopaedie. Cf. also Bachmann's Analytische Zahlentheorie, Chapter 11 to the end; and Hermite, Acta Math., 5; pp. 297–330, especially § V, also Math. Corresp. with Stieltjes.

Direct Sums, $\overline{gf}$.

2. Combining in $\overline{gf}'(n)$ those terms in which $d = a$;

$$\overline{gf}'(n) = \sum_{a=1}^{n} g(a)f'([n/a]). \tag{1}$$

Interchanging g, f, we have, since $\overline{gf} = \overline{fg}$:

$$\sum_{a=1}^{n} g(a)f'([n/a]) = \sum_{a=1}^{n} f(a)g'([n/a]), \tag{2}$$

which was first stated in a different form by Liouville. For $g = u_0$, (2) becomes Dirichlet's:

$$\sum_{a=1}^{n} f'([n/a]) = \sum_{a=1}^{n} [n/a]f(a); \tag{3}$$

which, with (1) for $g = u_0$, gives the useful formulæ:

$$\overline{u_0 f'}(n) \equiv \bar{f}'(n) = \sum_{a=1}^{n} [n/a]\, f(a) = \sum_{a=1}^{n} f'([n/a]). \tag{4}$$

3. **Illustrations of § 2 for specific ψ's.** When f, g are ψ's, it is always possible to express $\overline{fg}$, $\bar{f}$ each as a single ψ, in general simpler than $\overline{fg}$, $\bar{f}$. We shall use such expressions* in illustrating § 2 for the following ψ's:

(i) $k_r(n)$, $= 1$ or 0 according as n is or is not an rth power.

(ii) $\varpi(n)$, $= (-1)^{\lambda(n)}$, where $\lambda(n)$ is the *total* number of prime divisors (> 1), of n.

(iii) $\mu(n)$, $= 0$ if n is divisible by a square > 1, otherwise, $= \varpi(n)$.

(iv) $\varphi_r(n)$, $=$ the number of integers $\not> n^r$ that are divisible by the rth power of no prime divisor of n^r; $\varphi_1 \equiv \varphi$, the ordinary totient. For $f \equiv k_2$, $g(n) \equiv \mu^2(n)$, we have $\overline{fg} = u_0$, $f'(n) = [\sqrt{n}]$; hence by (1):

$$n = \sum_{a=1}^{n} [\sqrt{n/a}]\mu^2(a). \tag{5}$$

We have $\overline{\varpi} = k_2$; hence $\overline{\varpi}'(n) = [\sqrt{n}]$, and by (4):

$$[\sqrt{n}] = \sum_{a=1}^{n} [n/a]\varpi(a). \tag{6}$$

Also, $\mu(1) = 1$; $\bar{\mu}(n) = 0$, $n > 1$; hence (4) gives as a complement to (5):

$$1 = \sum_{a=1}^{n} [n/a]\mu(a) = \sum_{a=1}^{n} \mu'([n/a]). \tag{7}$$

* As $\overline{\varpi} = k_2$, in deriving (6). Liouville, J. des Math. (2) t. 2 (1857), in 4 articles, *Sur quelques fonctions numériques*, gave a great many results of this kind. Thus, his fourth formula, Art. I, is in our notation, $\overline{\varphi\zeta_0} = \zeta_1$. All those we quote are proved in § 7 of *An Arith. Theory of Certain Numer. Fns.*, Univ. of Washington Publications in Science I (1). Or, it is easy to verify them from first principles.

Similarly, from $\varphi_r = u_r$:

$$\sum_{a=1}^{n} a^r = \sum_{a=1}^{n} [n/a]\varphi_r(a) = \sum_{a=1}^{n} \varphi_r'([n/a]). \tag{8}$$

These suffice to show the method for direct sums. The next application has an independent interest.

4. **Determinant forms.** Writing (4) in the form

$$\sum_{a=1}^{n} [n/a] f(a) = \bar{f}'(n),$$

we find on putting successively $n = 1, 2, \cdots, n$, and solving the resulting equations for $f(n)$, since obviously the determinant of the system is 1:

$$f(n) = \begin{vmatrix} 1 & 0 & 0 & \cdots & \bar{f}'(1) \\ 2 & 1 & 0 & \cdots & \bar{f}'(2) \\ 3 & 1 & 1 & \cdots & \bar{f}'(3) \\ 4 & 2 & 1 & \cdots & \bar{f}'(4) \\ 5 & 2 & 1 & \cdots & \bar{f}'(5) \\ 6 & 3 & 2 & \cdots & \bar{f}'(6) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \tag{9}$$

where the $(k+1)$th column is k 0's followed by $(k+1)$ 1's, followed by $(k+1)$ 2's, $\cdots$, $(k = 0, 1, \cdots, n-2)$. Thus by (6), (7), (8), the value of the determinant* (4) is $\bar{\omega}(n)$, $\mu(n)$, $\varphi(n)$, when the last column is respectively $[\sqrt{1}], [\sqrt{2}], \cdots, [\sqrt{n}]$; $1, 1, \cdots, 1$; $1, 3, \cdots, \frac{1}{2}n(n+1)$.

(9) is a special case, for f or $g = u_0$, of the determinant for $g(n)$ or $f(n)$ similarly deduced from (1). For specific ψ's, these give a great many curious identities between determinants of integers, e. g., when the determinants are substituted for their equivalent ψ's in $\psi_i = \psi_j\psi_k$; or, for ψ in $\psi(mn) = \psi(m)\psi(n)$, m, n relatively prime. Again (5) gives a determinant for $\mu^2(n)$, which therefore is the square of that just found for $\mu(n)$, or:

$$\begin{vmatrix} [\sqrt{1}] & 0 & 0 & \cdots & 1 \\ [\sqrt{2}] & [\sqrt{1}] & 0 & \cdots & 2 \\ [\sqrt{3}] & [\sqrt{1}] & [\sqrt{1}] & \cdots & 3 \\ [\sqrt{4}] & [\sqrt{2}] & [\sqrt{1}] & \cdots & 4 \\ [\sqrt{5}] & [\sqrt{2}] & [\sqrt{1}] & \cdots & 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 2 & 1 & 0 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ 4 & 2 & 1 & \cdots & 1 \\ 5 & 2 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}^2. \tag{10}$$

* The φ-case is well known, and is due to J. Hammond; *cf.* Lucas, Théorie des Nombres, p. 406, equation 10.

Also, since $\mu(n) = \mu^3(n) = \mu^5(n) \cdots$, all odd powers of the first determinant in (10) are equal; similarly all even powers of the second. We pass to more general examples which are very rich in special applications.

5. **Enumerative* functions, Ω.** Let c denote a class of positive integers, and let $\Omega_c(n) = 1$ or 0 according as n is or is not a member of c; also let $N_c(n)$ denote the number of members of c that do not exceed n; then $N_c(n) - N_c(n-1) = \Omega_c(n)$, and $\overline{\Omega}_c(n)$ is the total number of divisors of n that are members of c. Writing now $(\Delta f)(n) \equiv f(n) - f(n-1)$, we have $(\Delta f)'(n) = f(n) - f(0)$; whence by (1):

$$g(\Delta f)'(n) = \sum_{a=1}^{n} g(a)(f([n/a]) - f(0)), \tag{11}$$

which, written in full for clearness, is (cf. § 1):

$$\sum_{a=1}^{n} \{\sum_{d} g(d)(f(a/d) - f(a/d-1))\} = \sum_{a=1}^{n} g(a)(f([n/a]) - f(0)): \tag{11a}$$

and this, for $g \equiv u_0$, becomes:

$$\sum_{a=1}^{n} \{\sum_{d} (f(d) - f(d-1))\} = \sum_{a=1}^{n} (f([n/a]) - f(0)); \tag{12}$$

whence, for $f \equiv N_c$, and applying (4) to $\overline{\Omega}_c'(n)$ $(N_c(0) \equiv 0)$:

$$\overline{\Omega}_c'(n) = \sum_{a=1}^{n} N_c([n/a]) = \sum_{a=1}^{n} [n/a]\Omega_c(a) = \sum_{a=1}^{n} \Omega_c'([n/a]). \tag{13}$$

A simple reduction of (12) by (4) gives readily its more useful form:

$$\bar{f}'(n) = \sum_{a=1}^{n} (f([n/a]) + [n/a]f(a-1) - f(0)). \tag{14}$$

Of the special cases of these we note from (13) (similarly to (9)):

$$\Omega_c(n) = \begin{vmatrix} 1 & 0 & 0 & \cdots & \Omega_c'(1) \\ 2 & 1 & 0 & \cdots & \overline{\Omega}_c'(2) \\ 3 & 1 & 1 & \cdots & \overline{\Omega}_c'(3) \\ 4 & 2 & 1 & \cdots & \overline{\Omega}_c'(4) \\ 5 & 2 & 1 & \cdots & \overline{\Omega}_c'(5) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}, \tag{15}$$

which gives interesting results if c is the class of all primes, or the class of all squares, etc. Or, combining (14) in turn with (6), (7) and (8) for $r > 1$, we get:

* Césaro's designation.

$$(16)\qquad \sum_{a=1}^{n}(f([n/a])+[n/a]f(a-1))=[\sqrt{n}],\ 1,\ \tfrac{1}{2}n(n+1),$$

according as $f=\overline{\omega},\ \mu,\ \varphi$. For c the class of all primes, and $P(n)$ the number of primes $\geqslant n$, we have $\overline{\Omega}_c(n)\equiv\lambda(n)$ (defined, § 3 (ii)); and (13) gives

$$\lambda'(n)=\sum_{a=1}^{n}P([n/a]).$$

As a last illustration of § 2, we shall derive two general symbolic formulæ* of wide applicability.

6. **Two symbolic formulæ.** Recalling that $u_r(n)\equiv n^r$, we write:

$$(17)\qquad f\equiv u_k;\qquad u_kg\equiv g_k;\qquad \text{hence}\qquad u_kg'=g_k'.$$

Let $x^k-(x-1)^k\equiv c_1x^{k-1}+c_2x^{k-2}+\cdots+c_k,\ k>0$: then, making the substitutions (17) in (11a) we get the first symbolic formula:

$$(18)\qquad \{g'^k-(g'-1)^k\}(n)=\sum_{a=1}^{n}[n/a]^kg(a),\qquad k>0,$$

where, after expansion, g'^r is to be replaced by g_r', viz., the left is $c_1g_{k-1}'(n)+c_2g_{k-2}'(n)+\cdots+c_kg_0'(n)$. The second symbolic formula is a consequence of (18):

$$(19)\qquad g'(g'+1)(g'+2)\cdots(g'+k)(n)=\frac{1}{k+2}\sum_{a=1}^{n}B_{k+2}(n/a)g(a),$$

$$k\geqslant 0,$$

where† $B_r(x)\equiv[x][x+1]\cdots[x+r-1]$, $=[x]([x]+1)\cdots([x]+r-1)$.

(19) is easily verified directly by means of (18) on expanding

$$g'(g'+1)\cdots(g'+k),\ \equiv g_{k+1}'+(k,1)g_k'+\cdots;$$

$$B_{k+2}(n/a)=[n/a]^{k+2}+(k+1,1)[n/a]^{k+1}+\cdots,$$

where (k,s) is the sum of all products s at a time of the numbers $1,\cdots,k$. Or, (19) follows without difficulty from (4). As examples, let $g\equiv u_0$; then we have $u_0u_k=\zeta_k$, where (v): $\zeta_k(n)=$ the sum of the kth powers of all divisors of n, whence $\zeta_0(n)=$ the number of divisors; then by (18), (19):

$$(20)\qquad \{\zeta'^k-(\zeta'-1)^k\}(n)=\sum_{a=1}^{n}[n/a]^k,$$

$$(21)\qquad \zeta'(\zeta'+1)\cdots(\zeta'+k)(n)=\frac{1}{k+2}\sum_{a=1}^{n}B_{k+2}(n/a).$$

* They relate to the symbolic calculus developed by Lucas (l. c., Chap. XIII) for investigating the properties of Bernoulli's numbers and allied topics. Each may be generalized to *arbitrary* (symbolic) functions, instead of symbolic $x^k-(x-1)^k$, etc.

† $B_r(x)=r!\,E_r(x)$, where $E_r(x)$ is the function considered by Hermite, l. c., § V.

Thus, putting $k = 3$ in (20), $k = 2$ in (21):

$$3\zeta_2'(n) - 3\zeta_1'(n) + \zeta_0'(n) = \sum_{a=1}^{n} [n/a]^3, \tag{20a}$$

$$\zeta_3'(n) + 3\zeta_2'(n) + 2\zeta_1'(n) = \frac{1}{4}\sum_{a=1}^{n} [n/a](1 + [n/a])(2 + [n/a])(3 + [n/a]). \tag{21a}$$

Of the ψ's already defined, φ_r (§ 3 (iv)) gives similar results, since we have $\varphi_0 = \varphi_k$, thus:

$$\{\varphi'^k - (\varphi' - 1)^k\}(n) = \sum_{a=1}^{n} [n/a]^k \mu(a); \text{ etc.} \tag{22}$$

We next consider the inversions* of (1), (4), which, applied to any result of the preceding kinds, give two more of the same nature, but (in general) for different functions.

Inverse Sums.

7. **Reciprocal functions.** The ψ which satisfies $\sum_n \psi(d) = 0$, $n > 1$, is the *unit* ψ. If $\overline{\psi_i \psi_j}$ is the unit ψ, ψ_i, ψ_j are *reciprocals* of each other. When ψ_i is given, its reciprocal exists and may easily be found.†

8. **Inversions of (1), (4).** Let $h(n) = 0$ for $n > N$, N a finite, positive integer, and consider

$$H(n) = \sum_{a=1}^{n} h(na) g(a). \tag{23}$$

Multiplying (23) by $g_1(n)$, and summing for $n = 1, 2, \cdots, N$:

$$\sum_{n=1} g_1(n) H(n) = \sum_{n=1} \sum_{a=1} h(na) g(a) g_1(n) \equiv \sum_{n=1} h(n) \overline{g g_1}(n). \tag{24}$$

We now restrict g_1 to be a ψ, and choose g_1 the reciprocal of g, so that $\overline{g g_1}$ is the unit (§ 7). Hence, in this case, (24) becomes, on transposing:

$$h(1) = \sum_{n=1}^{N} g_1(n) H(n). \tag{25}$$

If now in (1) we replace n by $[x/n]$, and write

$$\overline{gf}'([x/n]) \equiv H(n), \qquad f'([x/n]) \equiv h(n),$$

we get (23), and hence for $[x] = N$, by (25):

$$f'([x]) = \sum_{n=1}^{[x]} g_1(n) \overline{gf}'([x/n]);$$

* The process is an extension of the well-known method, given in Bachmann, l. c., pp. 310–315.

† Cf. Numerical Functions quoted in footnote 3. We shall assume such reciprocals as we use; they are taken from the reference cited; or it is easy to verify them à priori.

or, replacing the arbitrary positive integer $[x]$ by n, we find, as the inversion of (1):

$$f'(n) = \sum_{a=1}^{n} g_1(a)\overline{gf}'([n/a]). \tag{26}$$

We notice that (26) is significant only when g is a ψ, and hence the reciprocal g_1 (§ 7) exists. The reciprocal of u_0 is μ (defined, § 3 (iii)); hence, writing $g \equiv u_0$ in (26), we find, as the inversion of (4):

$$f'(n) = \sum_{a=1}^{n} \mu(a)\bar{f}'([n/a]), \tag{27}$$

which is significant for any f. Conversely, applying the same process of inversion to (26), (27), we get (1), (4); viz., the terms *direct* and *inverse*, applied to the sums we are considering, are interchangeable. If f, g are both ψ's, there is clearly, in addition to (26), another inversion of (1):

$$g'(n) = \sum_{a=1}^{n} f_1(a)\overline{gf}'([n/a]), \tag{28}$$

where f_1 is the reciprocal (§ 7) of f. It is unnecessary to illustrate (27); one example of (26), (28) shall suffice. Let $\theta(n)$ denote the number of decompositions of n into a pair of relatively prime factors; then we have $\mu^2 u_0 = \theta$, and putting $f \equiv u_0$, $g \equiv \mu^2$ in (1):

$$\theta'(n) = \sum_{a=1}^{n} u_0'([n/a])\mu^2(a) = \sum_{a=1}^{n} [n/a]\mu^2(a). \tag{29}$$

The reciprocal of $\mu^2(n)$ is $\overline{\omega}(n)$; that of u_0 is μ; hence (26), (28) give as inverses of (29):

$$n = \sum_{a=1}^{n} \overline{\omega}(a)\theta'([n/a]) = \sum_{a=1}^{n} \overline{\omega}'([n/a])\theta(a), \tag{30}$$

$$\sum_{a=1}^{n} \mu^2(a) = \sum_{a=1}^{n} \mu(a)\theta'([n/a]) = \sum_{a=1}^{n} \mu'([n/a])\theta(a). \tag{31}$$

(29) was otherwise found by Dirichlet. Enough examples have been given to show the use of the formulæ. On combining the principal formulæ with the results furnished by a detailed consideration of the reciprocals in § 7 (cf. footnote 9), we have an inexhaustible source of such identities; in particular, all the results of this same general kind in the references quoted may be derived with great ease.

SEATTLE, WASH.

SURFACES WHICH CAN BE GENERATED IN MORE THAN ONE WAY BY THE MOTION OF AN INVARIABLE CURVE.*

By Luther Pfahler Eisenhart.

The generation of a surface by the motion of an invariable curve is an intuitive conception. When the moving curve is a straight line, we have a ruled surface. When the motion is a rotation, a curve in a plane through the axis of rotation describes a surface of revolution. To this class of surfaces belong the sphere, the anchor ring and the quadrics of revolution. Another simple type is afforded by a surface generated by the translation of a curve C; it is called a surface of translation. The various curves described by points of C are congruent; any two of them can be brought into coincidence by translation. Hence a surface of translation possesses the property that it can be generated in two ways by the motion (in this case translations) of an invariable curve. Hyperboloids of one sheet and hyperbolic paraboloids are the only doubly ruled surfaces; on any one of them there lie two one-parameter families of real straight lines such that each line of one family meets all the lines of the other family, but none of its own family. Hence they also can be generated in two ways by the motion of a straight line. In this paper we discuss other types, not so simple nor so well known, of surfaces which can be generated in more than one way by the motion of an invariable curve.

1. **Double surfaces of translation.** An important example of surfaces generated in four ways by the motion of an invariable curve is afforded by the surfaces which are surfaces of translation in two ways. The equations of such a surface S are necessarily of the forms

$$x = X_1(x_1) + X_2(x_2) = -X_3(x_3) - X_4(x_4),$$
$$y = Y_1(x_1) + Y_2(x_2) = -Y_3(x_3) - Y_4(x_4),$$
$$z = Z_1(x_1) + Z_2(x_2) = -Z_3(x_3) - Z_4(x_4),$$

where x_1, x_2, x_3, x_4 are parameters. Incidentally we remark that when such a surface S is known, two others, S', S'', are known also; their equations are

* Address of the Vice-president and Chairman of Section A—Mathematics and Astronomy—American Association for the Advancement of Science, Pittsburgh, December 29, 1917.

$$x' = X_1(x_1) + X_3(x_3) = -X_2(x_2) - X_4(x_4),$$
$$y' = Y_1(x_1) + Y_3(x_3) = -Y_2(x_2) - Y_4(x_4),$$
$$z' = Z_1(x_1) + Z_3(x_3) = -Z_2(x_2) - Z_4(x_4),$$

and

$$x'' = X_1(x_1) + X_4(x_4) = -X_2(x_2) - X_3(x_3),$$
$$y'' = Y_1(x_1) + Y_4(x_4) = -Y_2(x_2) - Y_3(x_3),$$
$$z'' = Z_1(x_1) + Z_4(x_4) = -Z_2(x_2) - Z_3(x_3).$$

Lie* proposed the problem of finding all surfaces of this kind and obtained the general solution with the aid of Abel's theorem. Recently Darboux† obtained the same result in a simple and direct method without the use of Abel's theorem, which theorem follows therefore as a corollary. If $F(x, y) = 0$ defines any plane curve of the fourth order, and x_1, x_2, x_3 and x_4 are the abscissas of the points of intersection of the curve by a movable line, the functions X_i, Y_i, Z_i $(i = 1, 2, 3, 4)$ are given by

$$dX_i = \frac{dx_i}{\frac{\partial F}{\partial y_i}}, \qquad dY_i = \frac{x_i dx_i}{\frac{\partial F}{\partial y_i}}, \qquad dZ_i = \frac{y_i dx_i}{\frac{\partial F}{\partial y_i}}.$$

Hence X_i, Y_i and Z_i are the abelian integrals of the first kind relative to the four points of the curve situated on the same variable straight line.

We refer the reader to Darboux's paper for details, and content ourselves with the statement of the following additional properties. Thus Darboux has shown‡ that the tangents to the four curves C_i of coördinates (X_i, Y_i, Z_i) are parallel to the corresponding tangent planes to the surfaces S, S' and S''. He reëstablishes also the following theorem of Lie:

The lines drawn through a fixed point parallel to the tangents to any one of the curves C_i are the generators of an algebraic cone of the fourth order.

Lie§ considered the case when the quartic $F(x, y) = 0$ is decomposable into two conics. In this case one of the three surfaces, S, S' and S'', can be generated in an infinity of ways by the translation of a curve. Of this type are the minimal surface of Scherk, $e^z = \sin x \csc y$, and the minimal helicoid, $z = \tan^{-1}(y/x)$.

2. **Surfaces of Gambier.** Consider two right circular cylinders, F and R, of radii f and r respectively, placed with their axes vertical and in contact

* Archiv for Mathematik og Naturvidenskab. Also Poincaré, Journal de Mathématiques pures et appliquées, Ser. 5, vol. 1 (1895), pp. 219–314, and Bulletin de la Société mathématique de France, vol. 29 (1901), pp. 61–86.

† Leçons des Surfaces, Second Edition, Paris, 1914, pp. 151–161.

‡ L. c.

§ Geometrie der Berührungstransformationen, Leipzig, 1896, p. 409.

along a generator of each, internally or externally. Imagine a third cylinder A of radius a attached to R so that both A and R have a common axis. As R rolls, without slipping, over the fixed cylinder F, each point of A describes an epitrochoid or a hypotrochoid, according as R is exterior or interior to F. As the argument is the same in either case, we assume that R is exterior to F. Then all the epitrochoids are congruent curves. In fact, any one of them can be brought into coincidence with any other by a translation along the axis of F and a suitable rotation about this axis. If the axis of F is taken for the z-axis of coördinates, the projection of any one of these curves on the plane $z = 0$ is defined by equations of the form

$$(1)\qquad \begin{aligned} x &= (f + r)\cos\varphi - a\cos\left(\frac{f + r}{r}\varphi\right), \\ y &= (f + r)\sin\varphi - a\sin\left(\frac{f + r}{r}\varphi\right), \end{aligned}$$

where φ is the same for the curves described by points on the same generator of A. If we take any curve C on A, this curve describes a surface S upon which lies a family of congruent epitrochoids. Consequently S can be generated also by a suitable displacement of one of the latter curves, and therefore S is a surface of the type sought. These surfaces have been considered by Gambier.* We call them the *surfaces of Gambier*.

The epitrochoids considered above can be generated in a second manner, namely by taking three new cylinders F_1, R_1 and A_1, associated in a manner similar to the above, with their corresponding radii given by†

$$(2)\qquad f_1 = \frac{af}{r}, \qquad r_1 = \frac{a}{r}(f + r), \qquad a_1 = f + r.$$

The one-parameter family of epitrochoids lying on a surface of Gambier generated as above are described in like manner by the points of a curve C_1 lying on the cylinder A_1 as the latter moves with R_1 when it rolls over F_1. Hence we have the following theorem of Gambier:

A surface of Gambier can be generated in three different ways by the motion of an invariable curve.

When, in particular, $r = \frac{1}{2}f$ and $a = r$, the epitrochoids are straight lines passing through the axis of F. In this case the surface is a right conoid whose particular form is determined by C. Thus any right conoid, just like any cylinder, can be generated by the motion of a curve whose

* Comptes Rendus, vol. 158 (1914), p. 1155.

† Cf. Loria, Spezielle Algebraische und Transcendente Ebene Kurven, Leipzig (1902), p. 483.

points describe straight lines. We considered the converse problem in a recent paper* and established by analytical processes the theorem:

Cylinders and right conoids are the only ruled surfaces which can be generated by the motion of an invariable curve such that points of the curve describe generators in the motion; for the conoids these curves are sections of the surface by cylinders having the axis of the conoid for a generator.

The surfaces of Gambier afford an example of surfaces generated by the motion of an invariable curve whose points describe congruent curves. So far as we know, it has never been determined whether these are the only surfaces possessing this property.

When, as in the case of the right conoids, the ratio f/r is commensurable, the motion of the cylinders is algebraic (as follows from (1)) and an infinite number of algebraic surfaces can be generated. Gambier mentions, in particular, the case when $f = 2r$, $a \neq r$, and C is an ellipse. The surface is unicursal of the sixth degree, and is defined by

$$x = r\cos\varphi + a\cos\varphi_1, \qquad y = r\sin\varphi + a\sin\varphi_1,$$

$$z = l\sin(\varphi + \varphi_1) + l'\cos(\varphi + \varphi_1),$$

where l and l' are constants. The three series of equal curves are defined by $\varphi = \text{const.}$, $\varphi_1 = \text{const.}$, $\varphi + \varphi_1 = \text{const.}$; all are ellipses.

3. **The Bennett mechanism.** In the generation of certain surfaces of the kind sought use can be made of a simple new mechanism discovered in 1903 by G. T. Bennett of Emanuel College, Cambridge.† It consists of two pairs of equal rods, of lengths a and b. They are made into a skew quadrilateral $ABCD$ with

$$AB = CD = a, \qquad BC = AD = b.$$

The rods are fastened together by hinges at the vertices, and are perpendicular to the axis of the hinge at the point of junction, the axes of the hinges not being parallel but subject to the condition that the axes at A and C made the angles α and β respectively with the axis at B. Since the angles between the rods at A and C are equal, and likewise at B and D, we put

$$\theta = DAB = BCD, \qquad \varphi = ABC = CDA.$$

* Rendiconti del Circolo Matematico di Palermo, vol. 41 (1916), pp. 94–102.

† Engineering, vol. 76 (1903), p. 777. Darboux studied deformations of this skew quadrilateral in his memoir, Sur un problème relatif à la théorie des courbes gauches, Mémoires de l'Académie des Sciences, 1908. Borel also dealt with these quadrilaterals in his Mémoire sur les déplacements à trajectoires sphériques, Mémoires présentés par divers savants à l'Académie des Sciences, vol. 23 (1905). See also Bricard, Journal de l'École Polytechnique, Ser. 2, Cahier 11 (1906), pp. 1–93.

We remark that the axes at the extremities of any side lie in parallel planes normal to the side.

In order to find the relations between the quantities a, b, α, β, θ and φ, we express the conditions that the sum of the projections of the sides of the quadrilateral on each of three non-coplanar lines is zero. We choose as these lines the sides, AB and BC, and the axis at B, which evidently is normal to the plane ABC. In order to carry out this process more readily, we replace CD by its three components $a\cos\theta$, $a\sin\theta\cos\beta$, $a\sin\theta\sin\beta$, along BC, the perpendicular to BC in the plane ABC at C and the normal to ABC at C respectively. In like manner we replace AD by its components $b\cos\theta$, $b\sin\theta\cos\alpha$, $b\sin\theta\sin\alpha$ along AB, the perpendicular to AB in the plane ABC at A and the normal to ABC at A respectively. Then the conditions that the sum of the projections of the sides of the quadrilateral on the lines AB, BC and the axis at B are zero assume the respective forms

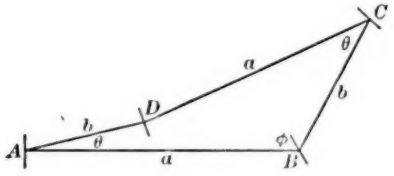

FIG. 1.

$$
\begin{aligned}
a - b\cos\varphi &= \cos\theta\,(b - a\cos\varphi) + a\sin\varphi\sin\theta\cos\beta,\\
b - a\cos\varphi &= \cos\theta\,(a - b\cos\varphi) + b\sin\varphi\sin\theta\cos\alpha,\\
a\sin\beta &= b\sin\alpha.
\end{aligned}
\tag{3}
$$

The last of these equations gives the only condition upon a, b, α and β. If $\sin\theta\sin\varphi$ is eliminated from the first two of the equations (3), the result is reducible to

$$\tan\frac{\theta}{2}\tan\frac{\varphi}{2} = \frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)} \equiv k. \tag{4}$$

Again, if a and b are eliminated from the first two of (3), the resulting equation is reducible to (4). Hence the three equations (3) are equivalent to their last and (4).

If we consider the sides BC and CD after the manner of AB and BC, we find that, in consequence of the third of (3), the axes at the extremities of CD make the angle α. In like manner the angle between the axes at

the extremities of DA is β. If the angle between the planes determined by a side of the quadrilateral and the two vertices not on the side is called a *dihedral angle* of the quadrilateral, from the preceding result and the third of (3) follows the theorem:

If one dihedral angle of a skew quadrilateral with opposite sides equal remains constant in the deformation of the quadrilateral, all the dihedral angles remain constant.

These results are true for any skew quadrilateral having opposite sides equal. For the Bennett mechanism α and β are constant, and consequently k also. Since θ and φ vary in accordance with the relation (4), the ratio of the angular velocities of two opposite rods relative to either of the other two rods is found by differentiating (4) with respect to t. This gives

$$\frac{\dfrac{d\theta}{dt}}{\dfrac{d\varphi}{dt}} = -\frac{\sin\theta}{\sin\varphi} = \frac{\cos\theta + 1}{\cos\varphi - 1}k = \frac{\cos\theta - 1}{\cos\varphi + 1}\frac{1}{k}. \tag{5}$$

Bennett has shown that the mutual rotation of the rods is illustrated by the rotation of the focal radii of an ellipse of eccentricity

$$e = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}. \tag{6}$$

In fact, using the polar equation of the ellipse in the forms

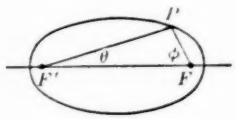

FIG. 2.

$$r_1 = \frac{l}{1 - e\cos\theta}, \qquad r_2 = \frac{l}{1 - e\cos\varphi},$$

and $r_1 + r_2 = 2a$, we get

$$\frac{2}{1 - e^2} = \frac{1}{1 - e\cos\theta} + \frac{1}{1 - e\cos\varphi}.$$

From this we find

$$\tan\frac{\theta}{2}\tan\frac{\varphi}{2} = \frac{1 - e}{1 + e} = k,$$

which is in keeping with (4) and (6).

From (4) it is seen that when θ or φ is zero, the other is π. In these cases, as follows from (5), the ratio of the angular velocities is greatest and least, namely k and $1/k$. The ratio is unity, when $\theta = \varphi$, that is when P is at the end of the minor axis of the ellipse.

4. **Doubly-decomposable systems of Koenigs.** Let two bodies A and C form a binary system of two parameters, that is, a system in which the relative positions of the two bodies depend upon two parameters. If there are two bodies B and D, such that the systems $\boxed{AB}$ and $\boxed{BC}$ depend upon one parameter each, both parameters being independent, and likewise the systems $\boxed{AD}$ and $\boxed{DC}$ depend upon one parameter each, the system $\boxed{AC}$ is said to be doubly decomposable, according to Koenigs.* If instead of two bodies B and D possessing this property, there are n such bodies, the system $\boxed{AC}$ is n-tuply decomposable.

Consider a point M of C. As C moves relatively to B, M describes a curve $\gamma_B{}^M$, and as B moves relatively to A this curve describes a surface $\Phi_A{}^M$. In like manner, as C moves relatively to D, M describes a curve

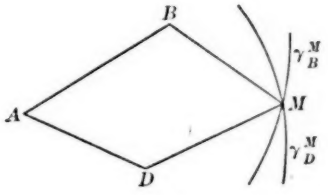

FIG. 3.

$\gamma_D{}^M$, which curve again describes $\Phi_A{}^M$, as D moves relatively to A. Hence we have the theorem of Koenigs.†

If $\boxed{AC}$ *is a doubly-decomposable system, a point of* C *in all motions of* C *relative to* A *describes a surface which is the locus of two families of equal curves.*‡

Evidently, if the system is n-tuply decomposable, the surface contains n families of equal curves.

We remark that Gambier has shown that the surfaces of Gambier can be generated in accordance with the above theorem of Koenigs. We shall not consider this case, but turn to an interesting example given by Koenigs,§ based on the Bennett mechanism.

5. **Hyperboloidal Gears.** Since the axes at the points A and B of the

* Comptes Rendus, vol. 117 (1913), p. 989.

† L. c.

‡ The only exception arises when $\gamma_B{}^M$ and $\gamma_D{}^M$ are superposed curves without interfering with the relative motion of B and D. In this case the curves are helices, circles or straight lines.

§ L. c.

Bennett mechanism are perpendicular to AB but inclined to one another at the angle α, two bodies Σ_A and Σ_B attached to these axes can be made to revolve about these respective axes in unison by means of hyperboloidal gears.* Two such gears meet along a common generator of two hyperboloids of revolution H_1 and H_2, with the axes at A and B of the Bennett mechanism. The radii r_1 and r_2 of the circles of gorge of the hyperboloids are such that their sum or difference is a, according as the hyperboloids roll externally to one another or one is internal to the other. If θ_1 and θ_2 denote the angles made by the common generator with the axes of H_1 and H_2, then from the geometrical interrelation of H_1 and H_2 we have

$$\frac{r_1}{r_2} = \frac{\tan \theta_1}{\tan \theta_2}. \tag{7}$$

If we assume that H_1 and H_2 are exterior to one another, and project the axes on the plane normal to AB and passing through the common generator, we get the situation illustrated by Fig. 4, where C is the point of

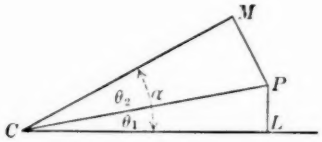

FIG. 4.

AB; L and M are projections of points on the axes of H_1 and H_2 respectively, and P is a point on the common generator. If ω_1 and ω_2 denote the angular velocities of H_1 and H_2 respectively, we have

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{PL}{PM} = \frac{\omega_2}{\omega_1}. \tag{8}$$

From this relation and $\alpha = \theta_1 + \theta_2$, we derive

$$\tan \theta_1 = \frac{\sin \alpha}{\frac{\omega_1}{\omega_2} + \cos \alpha}, \qquad \tan \theta_2 = \frac{\sin \alpha}{\frac{\omega_2}{\omega_1} + \cos \alpha}.$$

Combining this result with (7) and

$$r_1 + r_2 = a, \tag{9}$$

we get

* The following discussion of these gears is essentially the same as given by Reuleaux, Le Constructeur, Paris (1875), p. 463.

$$(10)\qquad \frac{r_1}{\omega_2^2 + \omega_1\omega_2 \cos\alpha} = \frac{r_2}{\omega_1^2 + \omega_1\omega_2 \cos\alpha} = \frac{a}{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos\alpha}.$$

When H_1 rolls inside of H_2, in comparison with the foregoing the ratio ω_1/ω_2 is negative, and so the analogous equations are in this case

$$(11)\qquad \begin{gathered} \frac{\sin\theta_1}{\sin\theta_2} = -\frac{\omega_2}{\omega_1}, \qquad \alpha = \theta_2 - \theta_1, \qquad r_2 - r_1 = a, \\ \tan\theta_1 = \frac{\sin\alpha}{\frac{\omega_1}{\omega_2} - \cos\alpha}, \qquad \tan\theta_2 = \frac{\sin\alpha}{-\frac{\omega_2}{\omega_1} + \cos\alpha}. \end{gathered}$$

6. **The Koenigs configuration.** We return to the consideration of the Bennett mechanism and place two bodies Σ_A and Σ_B on the axes at A and B, with hyperboloidal gears rolling externally to one another. We seek now to place a body Σ_C on the axis at C with a hyperboloidal gear rolling on H_2 and external to the latter. If θ_3 and r_3 are the angle and radius of the circle of gorge of the hyperboloid, H_3, of this gear, we must have

$$b = r_2 + r_3, \qquad \beta = \theta_2 + \theta_3.$$

When these values are substituted in the third of (3), the resulting equation is reducible, by means of (7) and a similar equation for r_2 and r_3, to

$$\cos\theta_1 = \cos\theta_3.$$

If $\theta_3 = \theta_1$, then $\alpha = \beta$, and consequently $a = b$. In this case we have a skew square, and by placing a gearing of angle θ_2 and radius r_2 on a body Σ_D revolving about the axis at D, we get a perfectly articulated system. The system $\Sigma_A\Sigma_C$ depends on two parameters, but the decomposition by means of Σ_B and Σ_D is similar. Hence we proceed to the other solution $\theta_3 = -\theta_1$.

This second solution leads to the case where H_3 lies within H_2. In fact, if, on the assumption $a > b$, we choose r_1 and r_2 so that

$$r_1 + r_2 = a, \qquad -r_1 + r_2 = b,$$

and take

$$\theta_1 = \frac{\alpha - \beta}{2}, \qquad \theta_2 = \frac{\alpha + \beta}{2},$$

in consequence of (3), equation (7) is satisfied. Then if we take for H_3 the hyperboloid H_1, and let it roll inside of H_2, equations analogous to (11) are satisfied. Furthermore, if we place a body Σ_D on the axis at D, and supply it with gears of the same form and size as H_2, with the under-

standing that the gear rolls externally to H_3 and about the gear H_1, we have a completely articulated system.

In this case the bodies $\Sigma_A\Sigma_C$ form a doubly-decomposable system, the rotation of the gears and the variation of the Bennett mechanism depending upon two independent parameters, such that Σ_B and Σ_D are the other bodies in the two decompositions. Hence a point of Σ_C, as the latter moves relative to Σ_A, generates a surface upon which there are two families of equal curves, namely skew epitrochoids.

It is readily shown that the line joining the mid-points of AC and BD is an axis of symmetry of the Bennett mechanism. It is also an axis of symmetry of the gears.

7. **Configuration of Bricard.** Another doubly-decomposable system which makes use of the Bennett mechanism has been discovered by Bricard.*

Given a triangle ABC and a point D' of its plane, and let the lengths of AD', BD' and CD' be denoted by l, m and n respectively. It is desired that a point D outside of the plane ABC be found so that in the faces ABD, ACD, BCD of the tetrahedron $ABCD$, points C', B', A' respectively can be found so that we have

$$\begin{aligned} A'D = B'C = C'B = D'A = l, \\ A'C = B'D = C'A = D'B = m, \\ A'B = B'A = C'D = D'C = n, \end{aligned} \tag{12}$$

and so that the tetrahedron $ABCD$ shall be inscribed in the tetrahedron $A'B'C'D'$, that is the point A is in the plane $B'C'D'$, B in the plane $A'C'D'$, and so on. Two tetrahedra each of which is inscribed in and circumscribed about the other were first considered by Möbius.† They are said to form a *Möbius configuration*. Bricard states, without proof, that this special Möbius configuration can be obtained as follows: Through D' draw any line L, and let α, β, γ be the points symmetric with respect to L of A, B, C respectively; then take

$$CD = A\beta = B\alpha, \qquad AD = B\gamma = C\beta, \qquad BD = A\gamma = C\alpha,$$

and construct the tetrahedron. In consequence of this choice it is possible to find points A', B', C', D' satisfying the conditions (12). The accompanying figures show how these points lie in the respective faces; their construction is unique and immediate. It remains to show that $ABCD$

* Comptes Rendus, vol. 158 (1914), p. 110.

† Crelle, vol. 3 (1828), pp. 273–278; also Gesammelte Werke, vol. 1, pp. 405–438.

is inscribed in $A'B'C'D'$. In order to do this we make use of Möbius's general methods in the following form.

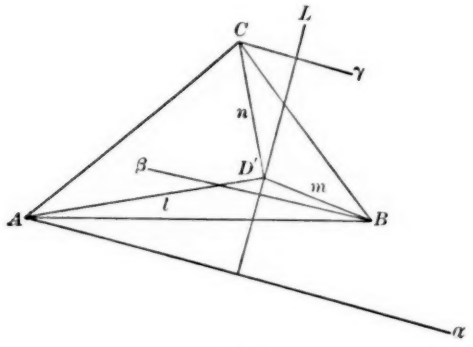

FIG. 5.

If A, B and C denote the distances of any three non-collinear points, A, B, C, from any line in their plane, and D' denotes the distance of the center of mass of three masses d_1, d_2, d_3 placed at A, B, C respectively, then

$$dD' = d_1A + d_2B + d_3C, \qquad d = d_1 + d_2 + d_3. \tag{13}$$

By taking moments about the sides of the triangle ABC, we find that D' is so placed that

$$d_1 : d_2 : d_3 = D'BC : D'CA : D'AB, \tag{14}$$

the terms on the right indicating areas of the respective triangles. It is evident then that if D' is any point in the plane of three points, A, B, C,

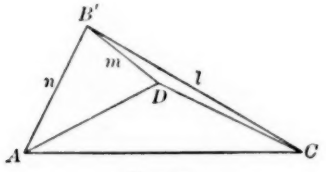

FIG. 6.

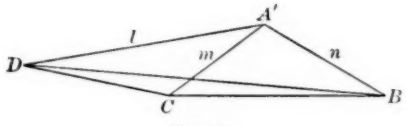

FIG. 7.

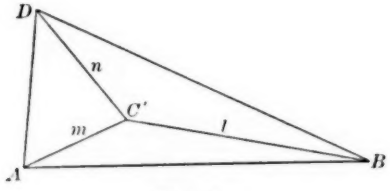

FIG. 8.

we have an equation of the form (13), it being understood that an area is positive or negative, according as the point D' lies on the same or opposite

side of a side of the triangle as the interior of the latter. Hence this equation gives analytical expression to the necessary and sufficient condition that a point D' lies in the plane of A, B, C.

Applying these considerations to the tetrahedra $ABCD$ and $A'B'C'D'$, we have, in addition to (13), the following equations:

$$\begin{aligned} aA' &= a_1B + a_2C + a_3D, \\ bB' &= b_1C + b_2D + b_3A, \\ cC' &= c_1D + c_2A + c_3B. \end{aligned} \tag{15}$$

If we require that A is to lie in the plane $B'C'D'$, a similar relation must hold. We write it in the form

$$a'A = a_1'bB' + a_2'cC' + a_3'dD',$$

since we wish to avoid fractional expressions, when in the right-hand member we substitute for B', C' and D' their expressions from (13) and (15). When this substitution is made, we get

$$(a_1'b_3 + a_2'c_2 + a_3'd_1 - a')A + (a_2'c_3 + a_3'd_2)B + (a_1'b_1 + a_3'd_3)C + (a_1'b_2 + a_2'c_1)D = 0.$$

Since A is not in the plane of B, C, D, we must have

$$a_2'c_3 + a_3'd_2 = 0, \qquad a_1'b_1 + a_3'd_3 = 0, \qquad a_1'b_2 + a_2'c_1 = 0. \tag{16}$$

In order that these equations be consistent, we must have

$$b_2c_3d_3 = -b_1c_1d_2. \tag{17}$$

When this is satisfied, we get the ratios $a_1' : a_2' : a_3'$ from (16).

In like manner the conditions that B, C and D lie in $A'C'D'$, $A'B'D'$ and $A'B'C'$ respectively are

$$\begin{aligned} a_2c_1d_1 &= -a_3c_2d_3, \\ a_1b_2d_1 &= -a_3b_3d_2, \\ a_1b_1c_2 &= -a_2b_3c_3. \end{aligned} \tag{18}$$

This result is due to Möbius,* who observed that if (17) and the second of (18) be multiplied together, we get

$$a_1b_1c_1d_1 = a_3b_3c_3d_3,$$

which follows also from the first and third of (18). Hence if three of the

* L. c.

vertices of $ABCD$ are inscribed in $A'B'C'D'$, so also is the fourth. For a general method of finding $A'B'C'D'$ for a given $ABCD$ we refer the reader to Möbius's article, and turn now to the proof that the tetrahedra of Bricard satisfy these conditions.

From Fig. 5 we find

$$a_1 : a_2 : a_3 = AD'\beta : -AD'\gamma : d_1,$$

$$b_1 : b_2 : b_3 = -BD'\gamma : d_2 : -AD'\beta,$$

$$c_1 : c_2 : c_3 = d_3 : AD'\gamma : BD'\gamma.$$

As these values satisfy (17) and (18), we have a configuration of Möbius.

In consequence of the relations (12), the six quadrilaterals $A'BC'D$, $B'CD'A$, $AC'BD'$, $A'CD'B$, $A'DB'C$, $AB'DC'$ are skew parallelograms. Together they form a kind of skew parallelepiped such that the three edges meeting in a vertex are coplanar. Hence Bricard calls it a *skew parallelepiped with flat trihedral angles*. If we imagine D' fixed, the possible deformations of the figure are accomplished by allowing A, B and C to move in circles of respective radii l, m and n about D' in their plane. Hence the deformation involves three parameters, when l, m and n are fixed.

If we require that $A'BC'D$ be deformed so that one of its dihedrals is constant (and consequently all are constant by the theorem of § 3), the deformations involve only two independent parameters. Consider the dihedral with edge BC'. Since C' is in the face ABD, and B is in the face $A'C'D'$, the angle of the planes ABD and $A'C'D'$ is, by hypothesis, constant in the deformation. Proceeding in like manner with the other three dihedrals of the skew parallelogram $A'BC'D$, we find that each of the following pairs of planes meet under angles constant in the deformation:

$$A'C'D',\ BCD; \qquad A'B'C',\ BCD; \qquad A'B'C',\ ABD.$$

The dihedral of the skew parallelogram $A'DB'C$ with the edge $A'D$ is the angle between $A'B'C'$ and BCD, which as we have just shown is constant. Hence all the dihedrals of $A'DB'C$ are constant. Proceeding in this way we show that *all six parallelograms of a skew parallelepiped with flat vertices have constant dihedrals, when a dihedral of any one parallelogram is constant.*

The foregoing result shows that there are deformable skew parallelepipeds whose six skew parallelograms form Bennett mechanisms. The axes of the mechanism $A'BC'D$ are the normals a', b, c', d to the faces

BCD, $A'C'D'$, ABD, $A'B'C'$ of the tetrahedra, at A', B, C' and D. In similar manner we get the axes of the other mechanisms.

An edge of a mechanism and its adjoining axes form an invariable system, and we can attach to it a body. Thus we attach to $A'D$ and its axes, a', d, a body $\Sigma_{a'd}$ and to BC' and its axes, b, c', a body $\Sigma_{bc'}$. These are articulated by means of the mechanism $A'BC'D$. By means of $D'BC'A$ we articulate $\Sigma_{ad'}$ with $\Sigma_{bc'}$. In like manner $\Sigma_{a'd}$ and $\Sigma_{ad'}$ are articulated with $\Sigma_{b'c}$ by means of $A'CB'D$ and $AB'CD'$. Hence the set $\Sigma_{a'd}$, $\Sigma_{ad'}$ forms a doubly-decomposable system. In like manner the skew parallelepiped defines five other doubly-decomposable systems, so that by this means we are able to generate surfaces of the kind sought, as described in § 4.

PRINCETON UNIVERSITY.

A CLASS OF DEVELOPMENTS IN ORTHOGONAL FUNCTIONS.

By Tomlinson Fort.

I.

1. Consider the differential equation

$$\frac{d}{dx}(k(x)y') + (\lambda^2 g(x) - h(x))y = 0; \tag{1}$$

where $k(x)$, $g(x)$ and $h(x)$ are real functions of the real variable x and where $g(x) > 0$ and $k(x) > 0$ have fourth derivatives and $h(x)$ a second derivative, limited and absolutely integrable in the Riemann sense over

$$a \leqq x \leqq b \tag{2}$$

and where moreover $g(a) = g(b)$, $g'(a) = g'(b)$, $k(a) = k(b)$ and $k'(a) = k'(b)$; subject to the conditions

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{3}$$

Denote the characteristic values for the system consisting of (1) and

$$y(a) = y(b) = 0 \tag{4}$$

by $\lambda_1^2 < \lambda_2^2 < \lambda_3^2 < \cdots$. As is well known* there exist an infinite set of values $l_1^2, l_2^2, l_3^2, \cdots$, satisfying the inequalities

$$l_1^2 < \lambda_1^2 < l_2^2 \leqq \lambda_2^2 \leqq l_3^2 < \lambda_3^2 < l_4^2 \leqq \lambda_4^2 \leqq l_5^2 < \cdots \tag{5}$$

such that when $\lambda^2 = l_j^2$, $j = 1, 2, 3, \cdots$, the system consisting of (1) and (3) is satisfied by at least one function, not identically zero. If $\lambda^2 = l_j^2 \neq l_{j-1}^2$ or l_{j+1}^2 all solutions of (1) satisfying (3) are linearly dependent. Denote by y_j a particular such solution, not identically zero. If $\lambda^2 = l_j^2 = l_{j+1}^2$ all solutions of (1) satisfy (3). Denote two particular linearly independent solutions by y_j and y_{j+1}.

If $l_j^2 \neq l_k^2$, y_j and y_k are orthogonal functions, that is they satisfy the relation

$$\int_a^b g(x)y_j(x)y_k(x)dx = 0. \tag{6}$$

* See, for example, Mason, Trans. Amer. Math. Soc., vol. VII, p. 360.

If $l_j^2 = l_{j+1}^2$ we choose y_j and y_{j+1} two solutions such that (6) holds* when k is replaced by $j + 1$.

We wish to consider the development of an arbitrary function, $f(x)$, in series of the form

$$c_1y_1 + c_2y_2 + c_3y_3 + \cdots, \tag{7}$$

where $c_1, c_2, c_3, \cdots$ are constants. If $f(x)$ is integrable from a to b the formal determination of the coefficients is immediate as in the case of Sturm-Liouville series

$$c_j = \frac{\int_a^b f(x)g(x)y_j(x)dx}{\int_a^b g(x)[y_j(x)]^2dx}. \tag{8}$$

2. In the present investigation we begin, as has generally been found necessary in such problems, by transforming† (1) to the form

$$\frac{d^2y}{dx^2} + (\lambda^2 - L(x))y = 0 \tag{9}$$

and for convenience the interval (2) into

$$0 \leqq x \leqq 2\pi. \tag{10}$$

It is deemed unnecessary to introduce new letters. Conditions (3) are transformed into

$$y(0) = y(2\pi), \qquad y'(0) = y'(2\pi), \tag{11}$$

and we denote by y_j, $j = 1, 2, \cdots$, the solution of (9) into which the old y_j is transformed. $L(x)$ has a limited integrable second derivative.

When $j \to \infty$, as is well known,‡ $\lambda_j^2 \to \infty$. Hence from (5) for large values of j, $l_j^2 > 0$. When this is the case let $l_j > 0$. Then for large values of j we can write*

$$y_j(x) = A \cos l_jx + B \sin l_jx + \frac{1}{l_j}\int_0^x L(\xi)y_j(\xi) \sin l_j(x - \xi)d\xi. \tag{12}$$

A and B are not both zero as otherwise we would have $y_j(0) = y_j'(0) = 0$ and hence $y \equiv 0$. We add arbitrarily the additional restriction,

$$A^2 + B^2 = 1. \tag{13}$$

We proceed from (12) to get an asymptotic expression for y_j as $j \to \infty$. To begin with, $|y_j|$ does not exceed a fixed positive number inde-

* An existence proof is the same as that given—Quart. Jour. of Pure and Ap. Math., vol. XLVI, p. 9.

† See, for example, Hobson, Proc. London Math. Soc., Series 2, vol. 6, p. 375.

‡ Ibid., p. 378.

pendent of x and j. A proof of this can be given exactly as is done for the corresponding theorem for Sturm-Liouville developments* and is consequently omitted.

Since y_j satisfies (11)

$$A = A\cos 2\pi l_j + B\sin 2\pi l_j + \frac{1}{l_j}\int_0^{2\pi} L(\xi)u_j(\xi)\sin l_j(2\pi - \xi)d\xi, \tag{14}$$

$$B = -A\sin 2\pi l_j + B\cos 2\pi l_j + \frac{1}{l_j}\int_0^{2\pi} L(\xi)u_j(\xi)\cos l_j(2\pi - \xi)d\xi. \tag{15}$$

Substitute from (12) in (14) and (15) and then again in the resulting equations. α will here, as in the remainder of this paper, denote a number independent of x which does not exceed in absolute value a positive number independent of j. $\alpha(x)$ will denote a function which never exceeds a fixed positive number independent of x and j. *α will generally denote quite different functions when used in different places*, not a particular function as would naturally be assumed. In addition let

$$\int_0^{2\pi} L(\xi)d\xi = 2N, \qquad l_j\int_0^{2\pi} L(\xi)\cos 2l_j\xi d\xi = 2K,$$

$$l_j\int_0^{2\pi} L(\xi)\sin 2l_j\xi d\xi = 2M \qquad \text{and} \qquad \tan l_j\pi = z.$$

We get after simplification

$$A\left(2l_j^2z^2 - 2l_jNz - 2Kz + M - Mz^2 + \frac{N^2}{2} - \frac{N^2}{2}z^2\right) = B(2l_j^2z + 2Mz - Nl_j - N^2z + Nl_jz^2 + K - Kz^2) + \frac{\alpha}{l_j}. \tag{16}$$

$$A(2l_j^2z - Nl_j + Nl_jz^2 - K + Kz^2 - 2Mz - N^2z) = B\left(-2l_j^2z^2 + M - Mz^2 + 2Nl_jz - \frac{N^2}{2} + \frac{N^2}{2}z^2 - 2Kz\right) + \frac{\alpha}{l_j}. \tag{17}$$

Suppose that as $j \to \infty$ l_jz does not remain finite then from (16) and (17)

$$2A + \beta_1 = \frac{2}{z}B + \beta_2, \qquad \frac{2}{z}A + \beta_3 = -2B + \beta_4,$$

where β_1, β_2, β_3 and β_4 are infinitesimal on some special set of values of λ or are zero. But these equations are impossible for values of A and B other than $A = B = 0$. Consequently l_jz remains finite as $j \to \infty$. The

* Ibid., p. 376.

elimination of A and B is now simple and yields the equation

$$\left[4l_j^4 - N^2l_j^2 - M^2 - K^2 + \frac{N^4}{4}\right]z^4 - 4l_j^3Nz^3 + \left[4l_j^4 - 2M^2 - 2K^2 + \frac{N^4}{2}\right]z^2 - 4l_j^3Nz + \left[N^2l_j^2 - K^2 - M^2 + \frac{N^4}{4}\right] = \frac{\alpha}{l_j}. \tag{18}$$

The left-hand member of this equation is devisable by $(1 + z^2)$. Consequently for the real values of z, in which alone we are interested,

$$z = \frac{2l_j^3N \pm \sqrt{4(M^2 + K^2)l_j^4 - \left(\frac{N^4}{4} - (M^2 + K^2)\right)^2}}{4l_j^4 - N^2l_j^2 - (M^2 + K^2) + \frac{N^4}{4}} + \frac{\alpha}{l_j^3}. \tag{19}$$

But $\lambda_{2n-1} \to n - \frac{1}{2}$ and $\lambda_{2n+1} \to n + \frac{1}{2}$ as $n \to \infty$.* Consequently from (5) and (19) $l_{2n} = n + (\alpha/n)$ and $l_{2n+1} = n + (\alpha/n)$ and consequently $K = (\alpha/n)$ and then

$$\begin{cases} \tan \pi l_{2n+1} = \dfrac{N}{2n} + \dfrac{M}{2n^2} + \dfrac{\alpha}{n^3}, \\ \tan \pi l_{2n} = \dfrac{N}{2n} - \dfrac{M}{2n^2} + \dfrac{\alpha}{n^3}, \end{cases} \tag{20}$$

and consequently

$$l_{2n} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2},$$

$$l_{2n+1} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2}.$$

Suppose $M \nrightarrow 0$ which is the case if $L(2\pi) \neq L(0)$. The substitution of these values in either (16) or (17) yields, giving a positive value to A which is allowable, $A = (1/\sqrt{2}) + (\alpha/n)$, $B = \pm(1/\sqrt{2}) + (\alpha/n)$. Repeated substitution in (12) yields

$$\begin{cases} u_{2n} = \dfrac{1}{\sqrt{2}}\cos l_{2n}x + \dfrac{1}{\sqrt{2}}\sin l_{2n}x + \alpha\dfrac{\cos l_{2n}x}{n} + \alpha\dfrac{\sin l_{2n}x}{n} + \dfrac{\alpha(x)}{n^2}, \dagger \\ u_{2n+1} = \dfrac{1}{\sqrt{2}}\cos l_{2n+1}x - \dfrac{1}{\sqrt{2}}\sin l_{2n+1}x + \alpha\dfrac{\cos l_{2n+1}x}{n} + \alpha\dfrac{\sin l_{2n+1}x}{n} + \dfrac{\alpha(x)}{n^2}. \end{cases} \tag{21}$$

* Ibid., p. 378.

† With the values obtained for A and B it is readily shown that

$$\int_0^x L(\xi)u_j(\xi)\sin l_j(x - \xi)d\xi = \frac{\alpha(x)}{l}.$$

In case $M \to 0$ a supplementary discussion is necessary. This discussion will be of a general character; in it further properties of the α's in (21) will be developed. Denote $\cos 2\pi l_j$ by c, $\sin 2\pi l_j$ by s, $\cos l_j\xi_i$ by c_i, $\sin l_j\xi_i$ by s_i, $\cos 2l_j\xi_i$ by C_i and $\sin 2l_j\xi_i$ by S_i, $i = 1, 2, 3, \cdots$. Let

$$\varphi_i = (sc_1 - cs_1)(s_1c_2 - c_1s_2)\cdots(s_{i-1}c_i - c_{i-1}s_i)(Ac_i + Bs_i),$$

$$\chi_i = (cc_1 + ss_1)(s_1c_2 - c_1s_2)\cdots(s_{i-1}c_i - c_{i-1}s_i)(Ac_i + Bs_i).$$

Substitution in (14) and (15) from (12) repeatedly yields

$$A = Ac + Bs + \frac{1}{l_j}\int_0^{2\pi} L(\xi_1)\varphi_1 d\xi_1 + \frac{1}{l_j^2}\int_0^{2\pi}\int_0^{\xi} L(\xi_1)L(\xi_2)\varphi_2 d\xi_2 d\xi_1$$

$$+ \cdots + \frac{1}{l_j^p}\int_0^{2\pi}\int_0^{\xi_1}\cdots\int_0^{\xi_{p-1}} L(\xi_1)L(\xi_2) \tag{22}$$

$$\cdots L(\xi_p)\varphi_p d\xi_p d\xi_{p-1}\cdots d\xi_1 + \cdots,$$

$$B = -As + Bc + \frac{1}{l_j}\int_0^{2\pi} L(\xi_1)\chi_1 d\xi_1$$

$$+ \frac{1}{l_j^2}\int_0^{2\pi}\int_0^{\xi_1} L(\xi_1)L(\xi_2)\chi_2 d\xi_2 d\xi_1 + \cdots + \frac{1}{l_j^p}\int_0^{2\pi}\int_0^{\xi_1} \tag{23}$$

$$\cdots\int_0^{\xi_{p-1}} L(\xi_1)L(\xi_2)\cdots L(\xi_p)\chi_p d\xi_p d\xi_{p-1}\cdots d\xi_1 + \cdots.$$

Consider now the expression φ_i, omitting the first and last factors, we have $(s_1c_2 - c_1s_2)\cdots(s_{i-1}c_i - c_{i-1}s_i)$ which we denote by ψ_i. If i is odd, $\psi_ic_1c_i$ is the same as $\psi_is_1s_i$ except that every s_k^2, $k = 1, \cdots, i$, is replaced by c_k^2 and vice versa. If i is even there is this same relation but for a change of sign. Replace now in $\psi_ic_1c_i$ and in $\psi_is_1s_i$ each s_k^2 by $(1 - C_k)/2$, each c_k^2 by $(1 + C_k)/2$ and each c_ks_k by $\frac{1}{2}S_k$. Multiply out all parentheses. $\psi_ic_1c_i$ and $\psi_is_1s_i$ now differ only in the signs of some of the terms, namely, those containing an odd number of C_k's. The same relation exists between $c_is_1\psi_i$ and $s_ic_1\psi_i$.

Now in (22) and (23) collect all the coefficients of As, Bs, Ac and Bc. We have absolutely convergent infinite series, as is seen by comparison with the series

$$\sum_{p=1}^{\infty}\frac{1}{p!}\left(\frac{2}{l_j}\int_0^{2\pi}|L(x)|dx\right)^p.$$

Let

$$\frac{s}{\cos^2 l_j\pi} = 2z \qquad \text{and} \qquad \frac{c}{\cos^2 l_j\pi} = 1 - z^2$$

as previously. The notation for the coefficients will be obvious. We get

$$(24)\quad \begin{cases} A(2l_j^2z^2 - 2l_j\bar{N}z - 2\bar{K}z + \bar{M} - \bar{M}z^2 + P - Pz^2) \\ \qquad = B(2l_j^2z - \bar{N}l_j + 2\bar{M}z - 2Pz + \bar{N}lz^2 + \bar{K} - \bar{K}z^2), \\ A(2l_j^2z - \bar{N}l_j + \bar{N}l_jz^2 - \bar{K} + \bar{K}z^2 - 2\bar{M}z - 2Pz) \\ \qquad = B(-2l_j^2z^2 + \bar{M} - \bar{M}z^2 + 2\bar{N}l_jz - P + \bar{P}z^2 - 2\bar{K}z). \end{cases}$$

The elimination of A and B and the solution of the consequent equation as in the case already discussed where $M \nrightarrow 0$ yields

$$(25)\quad z = \frac{4l_j^3\bar{N} \pm \sqrt{\begin{array}{l} l_j^4(16\bar{M}^2 + 16\bar{K}^2 - 16P^2 + 16N^2P - 4N^4) \\ \quad + 4l_j^2(\bar{M}^2 + \bar{K}^2 - P^2)(2\bar{N}^2 - 4P) - 4(P^2 - \bar{M}^2 - \bar{K}^2) \end{array}}}{8l_j^4 - 8l_j^2P + 4\bar{N}^2l_j^2 + 4P^2 - 4\bar{M}^2 - 4\bar{K}^2}.$$

Let us assume that $L(x)$ is analytic over (10), an assumption not made when $M \nrightarrow 0$.* In the latter case it was assumed only that $L^{IV}(x)$ was limited and absolutely integrable.

Let

$${}_qE_p^{(n)} = \int_0^{2\pi}\int_0^{\xi_1}\cdots\int_0^{\xi_{n-1}} L(\xi_1)L(\xi_2)\cdots L(\xi_n)c_{j_1}\cdots c_{j_p}s_{k_1}\cdots s_{k_q}d\xi_1d\xi_2\cdots d\xi_q,$$

where $j_1, \cdots, j_p, k_1, \cdots, k_q$ are distinct numbers of the succession $1, \cdots, n$ and $p + q \leqq n$. By integration by parts it is readily shown that ${}_qE_p^{(n)}$ is expansible in the form

$${}_qE_p^{(n)} = \frac{E_1}{l_j^{q+p}} + \frac{E_2}{l_j^{q+p+1}} + \cdots,$$

where $E_1, E_2, \cdots$, are polynomials in s and c.

Now write from (25), as is immediate,

$$z = \frac{z_1}{l_j} + \frac{\alpha}{l_j^2},$$

where z_1 is a constant. This result can be used to determine s and c of the form

$$s = \frac{s_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad c = 1 + \frac{c_1}{l_j} + \frac{\alpha}{l_j^2},$$

where s_1 and c_1 are constants. Consequently

$$\bar{M} = \frac{M_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad \bar{K} = \frac{K_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad \bar{N} = N + \frac{N_1}{l_j} + \frac{\alpha}{l_j^2},$$

$$P = \frac{N^2}{2} + \frac{P_1}{l_j} + \frac{\alpha}{l_j^2},$$

* It will appear that this is more than is generally necessary.

where M_1, K_1, N_1 and P_1 are constants. These values in (25) determine

$$z = \frac{z_1}{l_j} + \frac{z_2}{l_j^2} + \frac{\alpha}{l_j^3},$$

from which in a similar manner

$$\overline{M} = \frac{M_1}{l_j} + \frac{M_2}{l_j^2} + \frac{\alpha}{l_j^3},$$

where M_2 is also a constant; similarly for $\overline{K}$, $\overline{N}$ and P. These values in (25) then determine

$$z = \frac{z_1}{l_j} + \frac{z_2}{l_j^2} + \frac{z_3}{l_j^3} + \frac{\alpha}{l_j^4}, \qquad \text{etc.}$$

The process can be carried on as far as is desired. Suppose that at each stage of this process we substitute in the first of (24) and together with $A^2 + B^2 = 1$ solve for A and B, either A and B are indeterminate or are in the form, taking the positive sign with the radical

$$A = A_1 + \frac{a_1}{l_j} + \frac{\alpha}{l_j^2}, \qquad B = B_1 + \frac{b_1}{l_j} + \frac{\alpha}{l_j^2},$$

taking the negative sign

$$A = A_2 + \frac{a_2}{l_j} + \frac{\alpha}{l_j^2}, \qquad B = B_2 + \frac{b_2}{l_j} + \frac{\alpha}{l_j^2}.$$

A_1, A_2, B_1, B_2, a_1, a_2, b_1 and b_2 are fixed real numbers. If no matter how far the process just described is carried A and B remain indeterminate, equations (11) place no restriction on the solution and l_j is a double value. As previously

$$l_{2n} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2}, \qquad l_{2n+1} = n + \frac{N}{2n\pi} + \frac{\alpha}{n^2}.$$

We consequently can write

$$(26) \quad \begin{cases} y_{2n+1} = A_1 \cos l_{2n+1}x + B_1 \sin l_{2n+1}x \\ \qquad + \dfrac{a_1 - B_1 N(x)}{n} \cos l_{2n+1}x + \dfrac{b_1 + A_1 N(x)}{n} \sin l_{2n+1}x + \dfrac{\alpha(x)}{n^2}, \\ y_{2n} = A_2 \cos l_{2n}x + B_2 \sin l_{2n}x \\ \qquad + \dfrac{a_2 - B_2 N(x)}{n} \cos l_{2n}x + \dfrac{b_2 + A_2 N(x)}{n} \sin l_{2n}x + \dfrac{\alpha(x)}{n^2}. \end{cases}$$

which are the desired asymptotic forms, $2N(x) = \int_a^x L(x)d(x) \cdot y_{2n}$ and

y_{2n+1} are orthogonal, which fact together with (13) necessitates

$$(27)\quad \begin{cases} A_i^2 + B_i^2 = 1, \ (i = 1, 2), \\ A_1A_2 + B_1B_2 = A_1B_1 + A_2B_2 = 0, \quad A_1^2 + A_2^2 = B_1^2 + B_2^2 = 1, \\ A_1a_1 + B_1b_1 = A_2a_2 + B_2b_2 = A_1a_1 + A_2a_2 = B_1b_1 + B_2b_2 = 0, \\ A_1a_2 + A_2a_1 + B_1b_2 + B_2b_1 = 0, \\ A_1b_1 + B_1a_1 + A_2b_2 + B_2a_2 = 0. \end{cases}$$

3. Adopt the notation of (26) even in case $M \nrightarrow 0$, for then simply

$$A_1 = A_2 = \frac{1}{\sqrt{2}}, \qquad B_1 = -B_2 = \frac{1}{\sqrt{2}}.$$

Let $v_{2n+1} = A_1 \cos nx + B_1 \sin nx$, $v_{2n} = A_2 \cos nx + B_2 \sin nx$. Let $f(x)$ be an integrable function and let

$$\gamma_{2n+1} = \frac{\int_0^{2\pi} f(x)v_{2n+1}dx}{\int_0^{2\pi} (v_{2n+1})^2dx}, \qquad \gamma_{2n} = \frac{\int_0^{2\pi} f(x)v_{2n}dx}{\int_0^{2\pi} (v_{2n})^2dx}.$$

It can immediately be verified by means of (27) that

$$\gamma_{2n}v_{2n} + \gamma_{2n+1}v_{2n+1} = a_n \cos nx + b_n \sin nx,$$

where a_n and b_n are the Fourier constants for $f(x)$. Moreover, replacing l_{2n+1} and l_{2n} by the values obtained for them in terms of n and letting

$$c_j = \frac{\int_0^{2\pi} f(x)y_j(x)dx}{\int_0^{2\pi} [y_j(x)]^2dx}$$

from 26

$$(28)\qquad c_{2n}y_{2n} + c_{2n+1}y_{2n+1} = a_n \cos nx + b_n \sin nx + p_n,$$

where by (27)

$$p_n = \alpha\frac{a_n}{n^2} + \alpha\frac{b_n}{n^2} + \frac{\alpha}{n^2}\int_0^{2\pi} f(\xi)\alpha(\xi)d\xi.$$

In case that A_1, B_1, A_2 and B_2 are arbitrary but for (13), that is, where $l_j = l_{2n} = l_{2n+1}$, a double value, we choose for definiteness $A_1 = B_2 = 1$, $A_2 = B_1 = 0$ and as a result still have (28).

Let

$$s_{2n} = \sum_{n=1}^{n} c_{2n}y_{2n} + c_{2n+1}y_{2n+1}$$

and σ_{2n} the sum of the first $2n-1$ terms of the Fourier series for $f(x)$. The series $p_0 + p_1 + \cdots$ converges uniformly over $0 \leqq x \leqq 2\pi$. *Then* $s_{2n} - \sigma_{2n}$ *converges uniformly in* x *over* $0 \leqq x \leqq 2\pi$.

Assume first that $f(x)$ is analytic when $0 \leqq x \leqq 2\pi$. Both s_n and σ_n converge to $f(x)$ when $0 < x < 2\pi$.* But $s_{2n} - \sigma_{2n}$ is continuous and converges uniformly. Consequently $s_{2n} - \sigma_{2n}$ converges uniformly to zero over the closed interval $0 \leqq x \leqq 2\pi$.

Now let

$$s_{2n}(x) - \sigma_{2n}(x) = \int_0^{2\pi} f(\xi)\Phi_n(x, \xi)d\xi,$$

$$\Phi_n(x, \xi) = \frac{\alpha(x, \xi)}{n^2},$$

where $\alpha(x, \xi)$ for all values of n remains in absolute value less than a fixed positive number independent of x and ξ. It results that $\Phi_n(x, \xi)$ remains in absolute value less than a fixed number independent of x, n and ξ. No assumptions as to the character of $f(x)$ other than integrability are made here.

Next let $f(x)$ be continuous, $0 \leqq x \leqq 2\pi$, but not necessarily analytic. Form a sequence of analytic functions, $f_1, f_2, f_3, \cdots$, approaching f uniformly, $0 \leqq x \leqq 2\pi$. Obviously

$$\int_0^{2\pi} f(\xi)\Phi_n(x, \xi)d\xi = \int_0^{2\pi} (f(\xi) - f_n(\xi))\Phi_m(x, \xi)d\xi + \int_0^{2\pi} f_n(\xi)\Phi_m(x, \xi)d\xi.$$

Let η be arbitrarily small. Since Φ_n is bounded we can choose an $\bar{n}$ such that when $n \geqq \bar{n}$ the first integral is in absolute value less than $\frac{1}{2}\eta$. Having chosen $\bar{n}$ choose $\bar{m}$ so that when $m \geqq \bar{m}$ the last integral is in absolute value less than $\frac{1}{2}\eta$. Consequently $|s_{2n} - \sigma_{2n}| < \eta$. Consequently $s_{2n} - \sigma_{2n}$ converges to zero uniformly over the closed interval $0 \leqq x \leqq 2\pi$.

Extension to discontinuous functions is made by setting up a sequence of continuous functions, $f_1, f_2, \cdots$, such that

$$\int_0^{2\pi} |f_n(\xi) - f(\xi)|d\xi \to 0$$

and reasoning much as before.† *The result is obtained that* $s_{2n} - \sigma_{2n} \to 0$ *uniformly when* $0 \leqq x \leqq 2\pi$.

The following theorem is readily concluded.

* A. C. Dixon, Proc. L. Math. Soc., 1905, p. 99.

† See Haar, Math. Ann., Bd. 69, S. 355, where similar reasoning is carried through for the Sturm-Liouville and cosine series. See also Hobson, Theory of Functions of a Real Variable, p. 532, for classification of functions defined by sequences.

If $f(x)$ is a function integrable in the Lebesgue sense from 0 to 2π. The development $\sum_{\infty}^{n=1}(c_{2n}y_{2n} + c_{2n+1}y_{2n+1})$ *converges at any particular point of* $0 \leqq x \leqq 2\pi$ *when and only when the Fourier's series for $f(x)$ converges at that point and to the same value. It converges uniformly over the whole interval or any subinterval when and only when this is true of the Fourier's series and diverges to ∞ or $-\infty$ when and only when the Fourier's series does. Moreover it is summable by the method of the arithmetic mean when and only when the same thing is true of the Fourier's series and to the same value.*

It is to be noted that results from this theorem are immediately transferable to series (7) over the interval (2).*

II.

4. Very similar to the problem just discussed in I is that of developing a function in terms of the infinite succession of solutions of (9) satisfying

$$y(0) = -y(2\pi), \qquad y'(0) = -y'(2\pi). \tag{29}$$

The work throughout is the same as that gone through in I except for the details of the algebra which fact apparently makes a proof by the method of this paper to cover both cases impractical. The results will, however, be readily surmised and I shall give them without anything resembling a detailed proof.

Let the characteristic values be denoted by $l_j'^2$, $j = 1, 2, 3, \cdots$, and for large values of j let $z = \cot l_j'\pi$. Analogous to equations (24) we have

$$\begin{aligned} &A(2l_j'^2z^2 + 2l_j'\overline{N}'z + 2\overline{K}'z - \overline{M}'z^2 + \overline{M}' - P'z^2 + P') \\ &\qquad = B(-2l_j'^2z - 2\overline{M}'z - \overline{K}'z^2 + \overline{K}' + \overline{N}'l_jz^2 - \overline{N}'l_j + 2P'z), \\ &A(2l_j'^2z - \overline{N}'l_j'z^2 + \overline{N}'l_j' - 2\overline{M}'z - \overline{K}'z^2 + \overline{K}' - 2P'z) \\ &\qquad = B(2l_j'^2z^2 - 2\overline{K}'z + \overline{M}'z^2 - \overline{M}' + 2\overline{N}'l_jz - P'z^2 + P'), \end{aligned}$$

where, in case $L(0) \neq L(2\pi)$, P' is to be replaced by $N^2/2$, the dashes dropped and α/n added as in (16), (17). These equations are solved for z and together with $A^2 + B^2 = 1$ for A and B as in I. It is found that $z \to 0$. Hence from the inequalities

$$l_1'^2 \leqq \lambda_1^2 \leqq l_2'^2 < \lambda_2^2 < l_3'^2 \leqq \lambda_3^2 \leqq l_4'^2 < \cdots,$$

$$l_{2n-1}' = \frac{2n-1}{2} + \frac{\alpha}{n} \qquad \text{and} \qquad l_{2n} = \frac{2n+1}{2} + \frac{\alpha}{n}.$$

* See Hobson, Proceeding L. Math. Soc., Series 2, vol. VI, p. 387, where the same thing is done for Sturm-Liouville series.

Other work is a close parallel to that already carried through with the result that:

The development for $f(x)$ *in terms of solutions of* (9) *satisfying* (29) *bears the same relation to the development for* $f(x)$ *in terms of* $\sin [(2n+1)/2]x$ *and* $\cos [(2n+1)/2]x$, $n = 0, 1, 2, \cdots$, *as is borne by the series discussed in I to the Fourier's series for* $f(x)$.

May, 1916.

A FORMULA OF POLYNOMIAL INTERPOLATION.*

By Webster G. Simon.

Many proofs have been given of the following theorem of Weierstrass:†

Let $f(x)$ be a real function, defined and continuous for $a \leqq x \leqq b$; and let ϵ be an arbitrary positive constant. Then there exists a polynomial $G(x)$ such that for $a \leqq x \leqq b$

$$| f(x) - G(x) | < \epsilon.$$

Of these proofs one of the simplest is due to Landau.‡ A proof will be proposed here, which, while closely related to Landau's, is perhaps more elementary, in that integrals are replaced by finite sums; and at the same time it retains something of the elegance for which Landau's proof is conspicuous above others that are more elementary still.§ The formula used here has some interest for its own sake, as a formula of interpolation, and as such may be compared, for example, with those of Borel,‖ Faber,¶ and Runge.**

Without loss of generality, we may assume that $0 < a < b < 1$, since a linear transformation can be performed on x, if necessary, without affecting the essential conditions of the problem. Let the definition of $f(x)$ be extended by setting $f(x) = f(a)$ for $0 \leqq x \leqq a$, $f(x) = f(b)$ for $b \leqq x \leqq 1$. Then $f(x)$ is defined and continuous for $0 \leqq x \leqq 1$. We shall now prove the

* This paper, as originally written, contained in addition to the polynomial formula given here a trigonometric formula of approximation. While this paper was in the hands of the editors, Kryloff published in Bulletin des sciences mathématiques, second series, vol. 41 (1917), pp. 309–320, a paper in which he establishes the validity of the same formula. So the reader is referred to Kryloff's article for the trigonometric formula.

† Weierstrass, "Über die analytische Darstellbarkeit sogennanter willkürlicher Functionen einer reellen Veränderlichen," Sitzungsberichte der Kgl. Preuss. Akademie der Wissenschaften, 1885, pp. 633–639, 789–805. "Über die analytische Darstellbarkeit sogennanter willkürlicher Functionen reeller Argumente," Werke, vol. 3 (1903), pp. 1–37.

‡ Landau, "Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion," Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 337–345.

§ Cf., e. g., Lebesgue, "Sur l'approximation des fonctions," Bulletin des sciences mathématiques, series 2, vol. 22 (1898), pp. 278–287.

‖ Borel, Leçons sur les fonctions de variables réelles et les développements en séries de polynomes. Paris, 1905, pp. 79–82.

¶ Faber, "Über stets konvergente Interpolationsformeln," Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 19 (1910), pp. 142–146.

** Runge, Theorie und Praxis der Reihen, pp. 134–142.

THEOREM. *When n increases indefinitely, the polynomial in x (of degree $2n$)*

$$\sigma_n(x) = \frac{\sum_{i=0}^{n} f(t_i)[1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n}, \tag{1}$$

where $t_i = i/n$, approaches the limit $f(x)$ uniformly for $a \leqq x \leqq b$.

First consider the case when $f(x)$ is identically equal to *unity*, and write

$$\tau_n(x) = \frac{\sum_{i=0}^{n} [1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n}, \tag{2}$$

In this expression, the numerator is less than the denominator. For each of the two largest terms of the numerator is less than or equal to unity, each of the two next in order of magnitude is less than or equal to $(1 - t_1^2)^n$, and so on, while in the denominator there will be a surplus of terms which will not be called into comparison in this way at all. Therefore

$$\tau_n(x) < 1. \tag{3}$$

On the other hand, each of the two largest terms of the numerator of (2) is at least equal to $(1 - t_1^2)^n$, each of the next two in order of magnitude is at least equal to $(1 - t_2^2)^n$, and so on for at least k pairs of terms, if x is in the interval (a, b) and k, depending on n, is the greatest integer such that k/n is less than γ, γ being a positive quantity such that $0 < a - \gamma < b + \gamma < 1$. Then

$$\tau_n(x) > \frac{2\sum_{i=1}^{k}(1 - t_i^2)^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n} = 1 - \frac{1 + \sum_{i=k+1}^{n}(1 - t_i^2)^n}{\sum_{i=0}^{n}(1 - t_i^2)^n}.$$

For $i > k$, $1 - t_i^2 < 1 - \gamma^2$. Therefore

$$\tau_n(x) > 1 - \frac{1 + (n - k)(1 - \gamma^2)^n}{\sum_{i=0}^{n}(1 - t_i^2)^n} > 1 - \frac{1 + n(1 - \gamma^2)^n}{\sum_{i=0}^{n}(1 - t_i^2)^n}. \tag{4}$$

Since evidently $0 < 1 - \gamma^2 < 1$, $n(1 - \gamma^2)^n$ approaches zero as a limit as n increases indefinitely. Now

$$\lim_{n\to\infty} \sum_{i=0}^{n}(1 - t_i^2)^n = \lim_{n\to\infty} \sum_{i=0}^{n}\left(1 - \frac{i^2}{n^2}\right)^n = \infty,$$

as the following shows.

Out of $\sum_{i=0}^{n}(1 - i^2/n^2)^n$ consider only the first $[\sqrt{n}]$ terms.* Each of these terms will be decreased if n is put in place of i^2, so that

$$\sum_{i=0}^{n}(1 - i^2/n^2)^n > [\sqrt{n}](1 - 1/n)^n > K[\sqrt{n}],$$

where K is independent of n, since $(1 - 1/n)^n$ approaches a limit, not zero, as n increases indefinitely. Hence $\sum_{i=0}^{n}(1 - t_i^2)^n$ increases indefinitely with n.

On combination of this result with (4) and (3), it is seen at once that to every $\epsilon > 0$ there corresponds an N independent of x such that

$$1 - \epsilon < \tau_n(x) < 1, \qquad n \geqq N.$$

Therefore when $a \leqq x \leqq b$, $\tau_n(x)$ approaches unity uniformly as n increases indefinitely.

Now let $f(x)$ be the arbitrary function for which the theorem is stated. Let

$$\varphi_n(x) = f(x)\tau_n(x) = \frac{\sum_{i=0}^{n} f(x)[1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n}. \tag{5}$$

It is an immediate consequence of the preceding that $\varphi_n(x)$ converges uniformly to the limit $f(x)$. Thus, given a positive ϵ, it is possible to find an n_1, independent of x, such that

$$| \varphi_n(x) - f(x) | < \epsilon/3, \qquad n \geqq n_1. \tag{6}$$

On the other hand, by subtracting (5) from (1) and taking absolute values, we see that

$$| \sigma_n(x) - \varphi_n(x) | \leqq \frac{\sum_{i=0}^{n} | f(t_i) - f(x) | [1 - (t_i - x)^2]^n}{2\sum_{i=0}^{n}(1 - t_i^2)^n}. \tag{7}$$

Since $f(x)$ is continuous on the closed interval $0 \leqq x \leqq 1$, we can find a $\delta < 1$ such that for any two values x', x'', in the interval considered,

$$| f(x'') - f(x') | < \epsilon/3, \text{ when } | x'' - x' | < \delta.$$

It may be assumed further that $\delta < \gamma$. The expression on the right of (7) can be broken up into two parts, as follows. Let the terms in the numerator for which $-\delta \leqq t_i - x \leqq \delta$ be added to form the numerator

* $[\sqrt{n}]$ stands for the greatest integer in $\sqrt{n}$.

of a new fraction Σ_n' with same denominator, and let the remaining ones be similarly used to form a fraction Σ_n''. It is assumed now that x is in the interval $a \leqq x \leqq b$. The sum Σ_n' will be increased if $\epsilon/3$ is put in place of $|f(t_i) - f(x)|$. The resulting expression with the factor $\epsilon/3$ removed contains only a part of the terms of $\tau_n(x)$, all of which are positive. Thus

$$\Sigma_n' < \epsilon/3,$$

in consequence of (3). In Σ_n'', $[1 - (t_i - x)^2]^n < (1 - \delta^2)^n < 1$, and replacing each difference $|f(t_i) - f(x)|$ by $2M$, where M is the maximum of the absolute value of $f(x)$, we obtain

$$\Sigma_n'' < \frac{2M(n+1)(1-\delta^2)^n}{2\sum_{i=0}^{n}(1-t_i^2)^n}.$$

The numerator approaches zero as n increases indefinitely, and the denominator becomes infinite. Then for a sufficiently large n_2, independent of x,

$$\Sigma_n'' < \epsilon/3, \qquad n \geqq n_2; \qquad \Sigma_n' + \Sigma_n'' < 2\epsilon/3, \qquad n \geqq n_2,$$

and

$$|\sigma_n(x) - \varphi_n(x)| < 2\epsilon/3, \qquad n \geqq n_2.$$

On combining this with (6), and taking n_0 equal to the greater of n_1 and n_2, we finally obtain

$$|\sigma_n(x) - f(x)| < \epsilon, \qquad n \geqq n_0.$$

Therefore $\sigma_n(x)$ approaches the limit $f(x)$ uniformly as n increases indefinitely.

CHICAGO, ILLINOIS.

PLANE NETS WITH EQUAL INVARIANTS.*

By Gabriel M. Green.

Two geometric characterizations of plane nets with equal Laplace invariants have been given by Koenigs. One of them, which identifies such a net as the projection, upon a plane, of the asymptotic net on a curved surface, has no analogue in the corresponding case of a conjugate net with equal invariants on a surface.* The other characterization, however, which involves certain conics connected with the net, is essentially the same for a plane net and for a conjugate net on a surface.† Recently, an elegant characterization of conjugate nets with equal invariants was given by Wilczynski,† according to which the developables of the congruence of lines joining the first and minus first Laplace transforms of the points of the surface correspond to a conjugate net on the surface. This characterization can not, of course, be applied to plane nets, because there is no relation in the plane to correspond to that of conjugate directions on a surface.

Another characterization of conjugate nets with equal invariants has been given by the present writer,‡ viz., that for such a net each line of the congruence of lines joining the first and minus first Laplace transforms is met in its focal points by the tangents to the curves which correspond to the developables of the congruence. It is our purpose to extend this characterization to planar nets. Naturally, we can not speak of the developables of a congruence of lines in a plane, since every one-parameter family of lines in a plane forms a developable; therefore our characterization can not be carried over to the plane case in its entirety.

Let

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3) \tag{1}$$

be the point equations, in homogeneous coördinates, of a net of curves in the plane. We suppose as usual that through every point y of a certain region of the plane pass two and only two curves of the net, the curves

* Presented to the American Mathematical Society, September 5, 1917.

† "The general theory of congruences," Transactions of the American Mathematical Society, vol. 16 (1915), pp. 311–327.

‡ "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface," American Journal of Mathematics, vol. 38 (1916), pp. 287–324. Cf. in particular p. 318. Cf. also a note on this theorem in the Bulletin of the American Mathematical Society, vol. 24 (1918), pp. 221–225.

u = const. and v = const., and we shall suppose moreover that these curves have distinct tangents at y. Then the three functions y are such that the determinant

$$\begin{vmatrix} y_u^{(1)} & y_v^{(1)} & y^{(1)} \\ y_u^{(2)} & y_v^{(2)} & y^{(2)} \\ y_u^{(3)} & y_v^{(3)} & y^{(3)} \end{vmatrix}$$

is different from zero throughout the region under consideration, and the said three functions y are a fundamental set of solutions of a completely integrable system of partial differential equations of the form*

$$\begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a'y_u + b'y_v + c'y, \\ y_{vv} &= a''y_u + b''y_v + c''y. \end{aligned} \tag{2}$$

Any sufficiently regular one-parameter family of straight lines in the plane has an envelope. Under the suppositions already made, the tangents to the curves v = const., along a fixed curve u = const., have an envelope, and the point of this envelope which corresponds to y is

$$\rho = y_u - b'y. \tag{3}$$

Similarly, the tangents to the curves u = const. along a fixed curve v = const. have an envelope, and the point of this envelope corresponding to y is†

$$\sigma = y_v - a'y. \tag{4}$$

We shall call the line $\rho\sigma$ the *ray* of the point y, borrowing the term used by Wilczynski for the line joining the minus first and first Laplace transforms of a conjugate net on a curved surface. In the present case, the second of equations (2) is of the Laplace type, and ρ and σ are its minus first and first Laplace transforms respectively.

We shall denote by N the particular planar net under consideration. Let N' be any other net superposed on N. It may of course be defined by the equations

$$\varphi(u, v) = c_1, \qquad \psi(u, v) = c_2,$$

where the Jacobian of φ and ψ does not vanish anywhere in the region under consideration. Let t_1 and t_2 denote the tangents at y to the two curves of the net N, and t_1' and t_2' the tangents at y to the two curves of the net N'. The harmonic conjugate of t_1' with respect to t_1 and t_2 is a

* E. J. Wilczynski, "One-parameter families and nets of plane curves," Transactions of the American Mathematical Society, vol. 12 (1911), pp. 473–510.

† Cf. Wilczynski, loc. cit., p. 486.

definite line through y, say T_1, and the harmonic conjugate of t_2' with respect to t_1 and t_2 is a definite line T_2 through y and distinct from T_1.

Now allow the point y to trace out a curve C_1' of the net N'. Then the corresponding rays $\rho\sigma$ envelop a curve, and the ray $\rho\sigma$ which corresponds to a particular point y is tangent to this envelope at a particular point F_1. Similarly, the other curve C_2' of the net N' will yield a second point F_2 on the same line $\rho\sigma$. On each line $\rho\sigma$, we have therefore determined two points F_1 and F_2, which we shall call the *foci* or *focal points of the ray with respect to the net* N'. If t_1', t_2' are the tangents at y to the curves C_1', C_2' respectively, we shall say that F_1, F_2 correspond to t_1', t_2' respectively. Of course the net N' has thus far been quite arbitrary. We shall show presently that it is always possible to choose it in a unique way so that it shall have the following relation to the net N: *the lines* T_1 *and* T_2 *defined above are to meet the corresponding ray in the respective focal points* F_1 *and* F_2 *of the ray.* We shall say that the net N' is then *congruentially associated* with the net N, and shall refer to its defining property as the *congruential property*, because of the analogy to the case of the congruence of rays belonging to a conjugate net on a curved surface. The net N' is in this latter case the net of curves on the surface which correspond to the developables of the ray congruence.

The geometric characterization which we wish eventually to establish may now be stated as follows: *A necessary and sufficient condition that a planar net have equal invariants is that at each point of the net the tangents to the congruentially associated net meet the corresponding ray in the focal points of the ray, each tangent meeting the ray in the focal point which corresponds to the other.*

We shall first establish the existence of a unique net N' which is congruentially associated with the given net N. Let us suppose that a curve of the net N' is defined by means of the equations

$$u = u(t), \qquad v = v(t),$$

and let us denote the curve by C_t, putting in evidence the parameter t along the curve. If y is a point of the curve C_t, a point on the tangent to C_t at y is given by

$$y_u \frac{du}{dt} + y_v \frac{dv}{dt}.$$

That is, the tangent to C_t is the line joining y to this point. Of course, the actual values of du/dt and dv/dt are not significant in determining this tangent, but only the ratio dv/du, which we shall denote by μ.

As the point y traces out the curve C_t, the corresponding ray $\rho\sigma$ envelops a curve, and we wish to determine the corresponding point of

this curve. It is a point on the line $\rho\sigma$, and is therefore given by an expression of the form

$$R = \rho + \lambda\sigma,$$

where λ is to be determined so as to make R a focus of the ray. Let E_t denote the envelope; then the tangent to E_t is the line which joins the point R to the point

$$R_u \frac{du}{dt} + R_v \frac{dv}{dt},$$

or, what is the same thing, to the point $R_u + \mu R_v$, where $\mu = dv/du$. Now this line is to coincide with the line $\rho\sigma$, since $\rho\sigma$ is tangent to its envelope. Therefore we seek the condition that the three points ρ, σ, $R_u + \mu R_v$ lie on a line. We have, on differentiating (3) and (4) and using the values of the second derivatives of y taken from (2),

$$\rho_u = (a - b')y_u + by_v + (c - b_u')y, \qquad \rho_v = a'y_u + (c' - b_v')y,$$

$$\sigma_u = b'y_v + (c' - a_u')y, \qquad \sigma_v = a''y_u + (b'' - a')y_v + (c'' - a_v')y.$$

On replacing y_u and y_v by their values in terms of ρ, σ, and y, we obtain finally

$$\rho_u = (a - b')\rho + b\sigma + \mathfrak{C}y, \qquad \rho_v = a'\rho + Ky,$$

$$\sigma_u = b'\sigma + Hy, \qquad \sigma_v = a''\rho + (b'' - a')\sigma + \mathfrak{C}''y,$$

where

$$\mathfrak{C} = c + b'(a - b') + a'b - b_u', \qquad \mathfrak{C}'' = c'' + a'(b'' - a') + a''b' - a_v',$$

and where

$$H = c' + a'b' - a_u', \qquad K = c' + a'b' - b_v'$$

are the *Laplace-Darboux invariants* of the net. We may now calculate $R_u + \mu R_v$ in terms of ρ, σ, and y, and obtain without difficulty

$$R_u + \mu R_v = A\rho + B\sigma + (\mathfrak{C}''\lambda\mu + H\lambda + K\mu + \mathfrak{C})y,$$

where the values of A and B do not concern us. If this point is to lie on the line $\rho\sigma$, the coefficient of y must vanish, since y, ρ, σ can never lie on the same line; i. e.,

$$\mathfrak{C}''\lambda\mu + H\lambda + K\mu + \mathfrak{C} = 0. \tag{5}$$

Since μ is known for a given net N', the value of λ is found immediately from equation (5), and then the point $R = \rho + \lambda\sigma$ is the corresponding focus of the ray $\rho\sigma$. But we wish this point to coincide with the point in which $\rho\sigma$ is met by the line T which is the harmonic conjugate of the tangent to C_t with respect to the tangents of the net N, in order that N' may be congruentially associated with N. The condition for this is

that $\lambda = -\mu$. Substituting this value for λ in equation (5), and replacing μ by its value $dv\,du$, we find that *the differential equation of the net N' which is congruentially related to the given net N is*

$$\mathfrak{L}''du^2 + (H - K)dudv - \mathfrak{G}dv^2 = 0. \tag{6}$$

The process by which we obtained this equation shows that the net N' is unique. Now, if the net N has equal invariants, i. e., $H = K$, the middle term of equation (6) is absent. We may therefore state the theorem:

A necessary and sufficient condition that the net N have equal invariants is that the two tangents to the curves of the congruentially associated net N' at any point y separate harmonically the two tangents to N at y.

This characterization is of course equivalent to the one announced above, and reduces to it immediately when the definition of the congruentially associated net is applied.

A number of interesting questions suggest themselves in the present connection; an important one is, whether or not the relation between a net N and its congruentially associated net N' is a reciprocal one, and, if it is not, under what conditions it is. Again, the present writer's recent researches in the general theory of surfaces and rectilinear congruences* indicate a close analogy between the given net N and the asymptotic net on a curved surface, especially as regards the congruential property.

An isothermal planar net being an orthogonal net with equal Laplace-Darboux invariants, it is obvious that our theorem affords a purely geometric characterization of such nets. We have elsewhere given an analogous characterization of isothermal nets on a curved surface.†

HARVARD UNIVERSITY,
August 23, 1917.

* To appear in the Transactions of the American Mathematical Society.

† Transactions of the American Mathematical Society, vol. 18 (1917), pp. 480–488.

RECENT EXTENSIONS OF DESCARTES' RULE OF SIGNS.

By D. R. Curtiss.

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1. **Laguerre's proof. Extension to infinite series.** Descartes' Rule of Signs may be regarded both as regards its date of discovery* and in view of its simplicity as the first of that group of theorems which give evaluations for the number of real roots in a given interval of an algebraic equation with real coefficients. In its simplest form it may be thus stated:

An algebraic equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

with real coefficients cannot have more positive real roots than the sequence $a_0, a_1, \cdots, a_n$ *has variations of sign.*

As usually stated it also contains the conclusion as to negative roots, obtained by considering $f(-x)$. To Gauss† we owe the more precise statement that the number of variations of sign is either equal to the number of positive real roots, or else exceeds it by an even number.

Probably every reader is familiar with the proof, given more or less satisfactorily in elementary texts, which rests on the lemma that if a polynomial whose coefficients present r variations of sign is multiplied by $x-a$, where a is positive, the resulting polynomial will present at least $r+1$ variations. Another proof‡ rests on the division algorithm; if p is a positive real root, the quotient polynomial resulting from division of

* Descartes' Géometrie, 1637.

† Werke, vol. 3, p. 67.

‡ Encycl. der Math. Wiss., vol. 1, part 1, page 410. Note the erroneous inference that an equation without positive roots has no variations of sign in the sequence of its coefficients.

$f(x)$ by $x - p$ has at least one less variation of sign than $f(x)$. Into this and related questions we shall go more fully in § 8. Finally Descartes' Rule is often considered merely a corollary of the Budan-Fourier Theorem (see § 2).

A question which immediately presents itself is whether Descartes' Rule applies to infinite series. Obviously it can have no meaning where the number of variations of sign is infinite, and it could hardly be expected to give information as to roots of the function outside the interval of convergence of the series. A proof given by Laguerre* enables us very easily to answer this question, which could also be attacked with somewhat more difficulty by one of the other methods. This proof is the more interesting since it deduces Descartes' Rule from the more fundamental theorem of Rolle.

Following Laguerre, we establish the Rule of Signs by mathematical induction, supposing it true for polynomials presenting $(m - 1)$ variations of sign, and showing it must then be true for every polynomial with m variations. Obviously the rule is exact when there are no variations.

Let α be a real number. Then the equation

$$x^{-\alpha}f(x) = 0 \tag{1}$$

has the same positive roots as $f(x) = 0$. Hence by a corollary of Rolle's Theorem the derived equation of (1) has at least $r - 1$ positive roots if $f(x) = 0$ has r positive roots. But this derived equation, if we discard the factor $x^{-(\alpha+1)}$, is

$$xf'(x) - \alpha f(x) = 0, \tag{2}$$

in which the successive coefficients are

$$a_0(n - \alpha), \cdots, a_i(n - i - \alpha), \cdots, a_{n-1}(1 - \alpha), -a_n\alpha. \tag{3}$$

Let the sequence $a_0, a_1, \cdots, a_n$ present m variations of sign, one of these occurring between a coefficient a_k, and the next non-vanishing coefficient a_{k+l}, so that these coefficients are both not zero, and are of opposite sign. We now give to α a value between k and $k + l$. Then the sequence (3) will present the same number of variations of sign in its first k terms and in the remaining terms beginning with the $(k + l + 1)$th as do the corresponding terms of $a_0, a_1, \cdots, a_n$, but the variation between a_k and a_{k+l} is lost in (3). That is, (3) will present $m - 1$ variations of sign. Descartes' Rule is therefore applicable by hypothesis to (2). Hence

$$r - 1 \leqq m - 1,$$

* Oeuvres, vol. 1, pp. 3–5. Substantially this proof was given by de Gua in 1741. See Jensen, *Recherches sur la théorie des équations*, Acta Mathematica, vol. 36 (1912), p. 182.

from which results $r \leqq m$, as was to be proved. That $m - r$ is zero or an even integer follows from the fact that if m is odd a_0 and the last non-vanishing coefficient of $f(x) = 0$ must be of opposite sign; but this is a well-known condition that the equation have an odd number of positive roots. Similarly, if m is even r is even.

If we examine this proof closely we see that it applies not only to ordinary polynomials in x, but also to equations in which the variable occurs to fractional, or indeed, irrational powers. Further, *it applies to equations where $f(x)$ is an infinite series of powers of x arranged in ascending or descending sequence of exponents provided only that m is finite. The conclusion, however, applies only to the number of positive roots within the interval of convergence of the series, and, unless we strengthen our hypotheses, we cannot assert that $m - r$ is even.*

Thus if we take

$$f(x) = 2 - \frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 3} - \frac{x^3}{3 \cdot 4} \cdots,$$

we have $m = 1$, while $f(x) = 0$ has no real roots in the interval of convergence $[-1, +1]$.

If, however, we add certain hypotheses regarding the behavior of $f(x)$ at the ends (or merely at one end) of the interval of convergence, the parity of $m - r$ will follow. As an illustration, let $f(x)$ be expanded in a series of ascending positive integral powers of x having the interval of convergence $[-R, +R]$, and let the constant term be positive. Then if we make the hypothesis

$$\lim_{x=R} f(x) = +\infty \text{ or } -\infty,$$

we shall have $m - r = 0$ or an even positive integer. If, for example, m is odd, we shall have $\lim_{x=R} f(x) = -\infty$, and $f(R - \epsilon)$ will be negative. Since $f(0)$ and $f(R - \epsilon)$ are of opposite sign there must be an odd number of positive roots less than R, so that m and r are each odd, and their difference cannot be odd. Similarly if m is even.

The reader will readily supply corresponding hypotheses for such cases as that where $f(x)$ is a descending power series, or where it has an infinite number both of positive and negative powers.

2. **Descartes' Rule and the Budan-Fourier Theorem. Generalized Sturm functions.** Descartes' Rule is often treated as merely a special case of the Budan-Fourier Theorem, which is as follows:

BUDAN-FOURIER THEOREM.* *Let a and b be real numbers, $a < b$, and*

* This is very nearly the form given in Dickson's "Elementary Theory of Equations," p. 103, where the reader will find a careful proof.

$f(x) = 0$ an equation of degree n with real coefficients. Consider the sequence

$$f(x),\ f'(x),\ \cdots,\ f^{(n)}(x) \tag{1}$$

whose members are $f(x)$ and its successive derived functions, and let V_a denote the number of variations of sign in this sequence for $x = a$, vanishing terms being ignored. Then $V_a - V_b$ is either equal to r_{ab}, the number of real roots of $f(x) = 0$ in the interval $a < x \leqq b$, or exceeds r_{ab} by an even integer. A root of multiplicity k is here counted as k roots.

Descartes' Rule is obtained by setting $a = 0$ and $b = \infty$.

The proof of the Budan-Fourier Theorem sheds considerable light upon the *Cartesian Excess*, i. e., the number $m - r$ of § 1. If we restrict ourselves for simplicity to the case where none of the functions (1) vanishes at a or b and no consecutive two have a common root between a and b, the familiar proof gives us a condition that Descartes' Rule be exact. It is that *the positive real roots of each of the functions $f'(x), \cdots, f^{(n-1)}(x)$ give to the preceding and following functions of* (1) *opposite signs.* This will in particular be the case when the real roots of each of the functions (1) except the last separate those of the following in (a, b) from each other and from the ends of the interval.* We leave as a useful exercise for the reader the examination of cases where the equation $f(x) = 0$ or one or more of its derived equations has multiple roots.

The Budan-Fourier Theorem also gives another criterion for the exactness of the Rule of Signs. Since we have

$$V_a - V_c = (V_a - V_b) + (V_b - V_c),$$

$$r_{ac} = r_{ab} + r_{bc},$$

$$V_a - V_c \geqq r_{ac},$$

$$V_a - V_b \geqq r_{ab},$$

$$V_b - V_c \geqq r_{bc},$$

it follows at once that if the Budan-Fourier test is exact in an interval, finite or infinite, then it is exact in every sub-interval, and conversely. Since $V_\infty = 0$ and $V_{-\infty} = n$, the Budan-Fourier test is exact for every interval when and only when $f(x) = 0$ has all its roots real. As a corollary, we have the following:

Descartes' Rule is exact when and only when the Budan-Fourier test is exact in every interval $[a, b]$ where both a and b are positive. In particular it is exact for an equation having only real roots.

* In connection with this and the following, see Dickson's "Elementary Theory of Equations," problems 65–68, pages 175, 176.

The greater simplicity of the Budan-Fourier functions leads one to use them in preference to the Sturm functions in spite of the fact that test by the latter is always exact. The Sturm functions obtained from the modified Greatest Common Divisor algorithm are, however, special cases of the *generalized Sturm functions for an interval* $[a, b]$ defined as follows*:

A sequence of polynomials $f(x), f_1(x), \cdots, f_r(x)$ *is a set of generalized Sturm functions for the function* $f(x)$ *and the interval* $[a, b]$ *provided*:

(*a*) *No two consecutive functions vanish together at any point of* $[a, b]$;

(*b*) $f_r(x)$ *does not vanish in* $[a, b]$;

(*c*) *When, for* $1 \leqq i \leqq r - 1$, $f_i(x)$ *vanishes for a value* x_1 *in* $[a, b]$, $f_{i-1}(x_1)$ *and* $f_{i+1}(x_1)$ *have opposite signs*;

(*d*) *When* $f(x)$ *vanishes for a value* x_1 *in* $[a, b]$, $f_1(x_1)$ *has the same sign as* $f'(x_1)$.

These will be recognized as the properties of ordinary Sturm functions from which Sturm's Theorem is derived. We can now state in other terms the condition that the Rule of Signs be exact; it is that *the Budan-Fourier sequence* (1) *be a set of generalized Sturm functions for the interval* $[0, \infty]$.†

3. **Laguerre's problem.** To Laguerre we owe the development of the idea, in a series of extensions of the Rule of Signs, that if $\phi(x)$ is a polynomial which vanishes for no positive values of x, and if $f(x)$ does not verify the conditions for exactness, a proper choice of $\phi(x)$ may make the Cartesian excess less for the product $\phi(x)f(x)$, which has the same positive roots as $f(x)$, than for $f(x)$, and in fact may make the rule exact for $\phi(x)f(x)$. Here $\phi(x)$ or $f(x)$, or both, may be infinite series, the Rule of Signs then applying to the common interval of convergence. The resulting tests may be looked at as replacing the coefficients of $f(x)$ by sequences of rational functions of those coefficients. This is the class of extensions of Descartes' Rule which we shall now consider. A direct examination of the exactness of these tests would consist in showing that the Budan-Fourier functions for the product $\phi(x)f(x)$ verify the four parts of the definition of generalized Sturm functions. I am, however, aware of no case in which this method of attack has been used, and it seems to present considerable difficulties. In the following sections we shall use other methods.

4. **Polynomial multipliers for quadratic equations.** We proceed now to examine the case of a quadratic equation

$$f(x) \equiv x^2 - ax + b = 0. \tag{1}$$

* Dickson, l. c.; also Encycl. der Math. Wiss., vol. 1, part 1, page 417.

† As here stated, this calls for a definition of generalized Sturm functions which will cover the cases where (*a*) is not verified. This is left for the reader to do.

We wish to determine in all cases a polynomial *multiplier* $\phi(x)$ such that the product $\phi(x)f(x)$ presents no variations, one variation, or two variations of sign in its coefficients, according as $f(x) = 0$ has no positive roots, or one, or two.

We first note that we may take $\phi(x) \equiv 1$ in either of the latter two cases. In fact, if there is but one positive root either b must be negative, and $f(x)$ presents but one variation of sign, whether a is positive or negative, or else b is zero and a positive, so that the same conclusion follows. We can have two positive roots only when a and b are both positive, and $a^2 \geqq 4b$; here again the Rule of Signs is exact for $f(x)$ itself. It remains to examine the case where $a^2 < 4b$, and a and b are positive. A simple method of building up a product $\phi(x)f(x)$ which shall have all its coefficients positive is the following*: Multiply $x^2 - ax + b$ by $x^2 + ax + b$; the product will be $x^4 - a_1x^2 + b^2$, where $a_1 = a^2 - 2b$. If a_1 is negative we have already found our multiplier; otherwise we multiply $x^4 - a_1x^2 + b^2$ by $x^4 + a_1x^2 + b^2$, and continue the process until we obtain a product $x^{2^k} - a_kx^{2^{k-1}} + b^{2^k}$. It is always possible to obtain an a_k that is negative or zero, for if the roots of $f(x) = 0$ are

$$x_1 = \rho(\cos\theta + i\sin\theta), \qquad x_2 = \rho(\cos\theta - i\sin\theta),$$

then

$$(2) \qquad a = x_1 + x_2 = 2\rho\cos\theta, \qquad b = x_1x_2 = \rho^2, \qquad 0 < \theta < \frac{\pi}{2},$$

and we have

$$a_1 = a^2 - 2b = 4\rho^2\cos^2\theta - 2\rho^2 = 2\rho^2\cos 2\theta,$$

$$a_2 = a_1^2 - 2b^2 = 4\rho^4\cos^2 2\theta - 2\rho^4 = 2\rho^4\cos 4\theta,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_k = a_{k-1}^2 - 2b^{2^{k-1}} = 4\rho^{2^k}\cos^2 2^{k-1}\theta - 2\rho^{2^k} = 2\rho^{2^k}\cos 2^k\theta.$$

To make a_k negative, k is to be so chosen that $2^k\theta$ will be an angle of the second or third quadrant; that is, we may take k as the least positive integer for which

$$(3) \qquad 2^k\theta \geqq \frac{\pi}{2}, \qquad \text{or} \qquad \theta \geqq \frac{\pi}{2^{k+1}}.$$

We have thus obtained a *Cartesian multiplier*

$$(4) \qquad \phi(x) = (x^2 + ax + b)(x^4 + a_1x^2 + b^2)\cdots(x^{2^k} + a_{2^{k-1}} + b^{2^{k-1}}),$$

all of whose terms are positive.

We shall see later that a Cartesian multiplier may have negative terms, but those with positive terms most naturally suggest themselves. The

*See "The Degree of a Cartesian Multiplier," Bulletin of the American Mathematical Society, vol. 20 (1913), p. 21.

multiplier (4) is, in general, of higher degree than some of other types. The whole subject of multipliers having positive terms, for a quadratic $f(x) = 0$ of the type we are considering, has been discussed by E. Meissner,* who has given it a most ingenious geometric interpretation, as follows: From the origin O draw OA to the right on the x axis, of any length desired. We now complete the polygonal line $OABC\cdots O$ (see Fig. 1) which returns to O and in which each successive side, taken in the

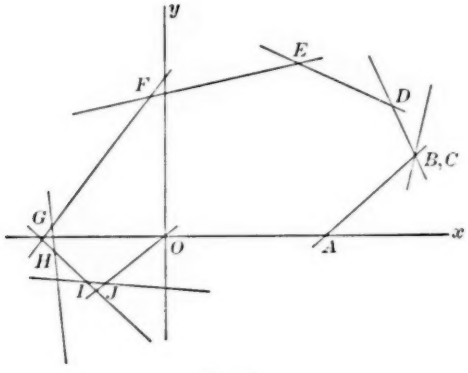

FIG. 1.

order OA, AB, etc., beginning with AB, makes the positive angle θ (where θ has the meaning given above) with its predecessor, the lines being directed as indicated by the order of the letters. Vertices may coincide, as B and C in Fig. 1, but the figure must be so drawn as to be concave towards O; i. e., as we proceed along each segment in the positive direction, O must lie to our left. If, now, we designate by a_0 the perpendicular distance from O to AB, and, in general, by a_k the perpendicular distance from O to the $(k+2)$th segment† of the polygonal line, then, if there are $m+3$ segments in all, the polynomial

$$\phi(x) = a_0 + \frac{a_1}{\rho}x + \frac{a_2}{\rho^2}x^2 + \cdots + \frac{a_m}{\rho^m}x^m \tag{5}$$

will be a Cartesian multiplier for $f(x)$.

To prove this, let us write

$$\phi(x)f(x) = \rho^2\left(a_0 + \frac{b_0}{\rho}x + \frac{b_1}{\rho^2}x^2 + \cdots + \frac{b_m}{\rho^{m+1}}x^{m+1} + \frac{a^m}{\rho^{m+2}}x^{m+2}\right), \tag{6}$$

* "Über positive Darstellungen von Polynomen," Math. Annalen, vol. 70 (1911), p. 223.

† We must here include in our count the r lines that must be drawn through a point where $r+1$ vertices coincide, giving r segments of zero length.

where, if we take $a_{-1} = a_{m+1} = 0$, we have

$$b_k = a_{k-1} + a_{k+1} - 2a_k \cos\theta, \qquad k = 0, 1, \cdots, m.$$

We wish to show that this expression is always positive.

By hypothesis, the perpendicular distance from the origin to the $(k+2)$th line of the figure is a_k, and the angle which its positive direction makes with the positive x-axis is $(k+1)\theta$. Hence its equation is, since the origin lies to the left of the positive direction,

$$x \sin(k+1)\theta - y\cos(k+1)\theta - a_k = 0.$$

Let (x_k, y_k) and (x_{k+1}, y_{k+1}) denote the intersections of this line with the preceding and the following one. Then we have

$$x_k = \frac{1}{\sin\theta}(a_k \cos k\theta - a_{k-1}\cos(k+1)\theta),$$

$$x_{k+1} = \frac{1}{\sin\theta}(a_{k+1}\cos(k+1)\theta - a_k \cos(k+2)\theta),$$

from which we obtain

$$\begin{aligned} x_{k+1} - x_k &= \frac{1}{\sin\theta}[(a_{k-1} + a_{k+1})\cos(k+1)\theta - a_k(\cos(k+2)\theta + \cos k\theta)] \\ &= \frac{1}{\sin\theta}[(a_{k-1} + a_{k+1})\cos(k+1)\theta - 2a_k\cos\theta\cos(k+1)\theta] \\ &= \frac{b_k}{\sin\theta}\cos(k+1)\theta. \end{aligned}$$

Since $x_{k+1} - x_k$ is the projection on the x-axis of a directed segment making the angle $(k+1)\theta$ with the positive x-axis, it follows that $b_k/\sin\theta$ is positive, and is the length of the $(k+2)$th segment of the polygonal line. Our proof that b_k is positive is now complete, as $\sin\theta$ is always positive.

Thus every construction of the type indicated corresponds to a Cartesian multiplier with positive or zero coefficients. A careful consideration of the proof just given will show that every such multiplier determines a diagram of the above type.

As a corollary, we easily deduce the minimum degree for a Cartesian multiplier, since the least number of segments of a polygonal line of the kind considered corresponds to an integer m such that

$$(m+1)\theta < \pi \leqq (m+2)\theta; \tag{7}$$

that is, m is the least positive integer such that $(m+2)\theta \geqq \pi$. A par-

ticularly interesting solution* is obtained by making m vertices coincide at A. Here a_0 may be chosen arbitrarily, since the length OA is at our choice, and we have

$$OA = \frac{a_k}{\sin (k+1)\theta}, \qquad k = 0, 1, 2, \cdots, m,$$

so that

$$a_k = \frac{a_0}{\sin \theta} \sin (k+1)\theta.$$

The product (6) then takes the form

$$\phi(x)f(x) = \rho^2 a_0 \left(1 - \frac{\sin (m+2)\theta}{\sin \theta} \cdot \frac{x^{m+1}}{\rho^{m+1}} + \frac{\sin (m+1)\theta}{\sin \theta} \frac{x^{m+2}}{\rho^{m+2}}\right).$$

This has at most three non-vanishing coefficients. In § 5 we shall investigate the general question, where the degree of $f(x)$ is n, of polynomial multipliers such that the product polynomial has at most $n+1$ non-vanishing coefficients. We shall prove that if there is any Cartesian multiplier of degree m, there is at least one of degree m of the type indicated.

We conclude this section by obtaining inequalities for the minimum degree of a Cartesian multiplier for the quadratic, expressed in terms of the coefficients. We restrict ourselves to the case of interest where $m \neq 0$, and hence $0 < \theta < (\pi/2)$. Here we have, by (7),

$$m + 1 < \frac{\pi}{\theta} < \frac{\pi}{\sin \theta}.$$

But

$$4b - a^2 = 4\rho^2 - 4\rho^2 \cos^2 \theta = 4\rho^2 \sin^2 \theta = 4b \sin^2 \theta,$$

so that we have the inequality

$$m + 1 < \frac{2\sqrt{b}}{\sqrt{4b - a^2}} \pi.$$

On the other hand

$$m + 2 \geqq \frac{\pi}{\theta} > \frac{\pi}{\tan \theta} = \frac{a}{\sqrt{4b - a^2}} \pi.$$

Thus we obtain the inequalities

$$(8) \qquad \frac{a}{\sqrt{4b - a^2}} \pi - 2 < m < \frac{2\sqrt{b}}{\sqrt{4b - a^2}} \pi - 1.$$

As an illustration take the quadratic

$$x^2 - 5x + 7 = 0.$$

* Ascribed by Meissner to A. Hurwitz.

The inequalities (8) show that m lies between 7 and 9, and hence must be equal to 8. The multiplier (4) is of degree 14.

5. **The existence of polynomial multipliers for the general algebraic equation with real coefficients.*** It is now comparatively easy to obtain a formula for a Cartesian multiplier of special type for the general algebraic equation $f(x) = 0$ of degree n with real coefficients. This, to be sure, is in terms of the roots. But in § 6 we shall show how to obtain multipliers that are rational in terms of the coefficients of $f(x)$, once we know that a multiplier exists of given degree.

Let $f(x) = 0$ have $2n_1$ imaginary roots, n_2 zero and negative roots, and n_3 positive roots so that

$$n = 2n_1 + n_2 + n_3. \tag{1}$$

We may then write

$$f(x) = p_{2n_1}(x)q_{n_2}(x)s_{n_3}(x), \tag{2}$$

where the factors are polynomials of the degrees indicated by the subscripts, and the first corresponds to the imaginary roots of $f(x) = 0$, the second to the zero and negative roots, and the third to the positive roots. Since the Rule of Signs is exact for an equation having only real roots, we will hereafter suppose that $n_1 > 0$.

We have shown in the preceding section that each quadratic factor of $p_{2n_1}(x)$ has a Cartesian multiplier; hence there is a polynomial $P(x)$ such that the product $P(x)p_{2n_1}(x)$ has all its coefficients positive. The coefficients of $q_{n_2}(x)$ are likewise all of one sign, which may be taken as positive, and the same must be true of the product $P(x)p_{2n_1}(x)q_{n_2}(x)$. Let the degree of this polynomial be N, an integer greater than 1, since $n_1 > 0$.

Since $(x - \alpha_i)$ is a factor of $(x^N - \alpha_i{}^N)$, there is a multiplier $S(x)$ for $s_{n_3}(x)$, which, if we write

$$s_{n_3}(x) \equiv k(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_{n_3}),$$

has a form, easily supplied by the reader, such that

$$S(x)s_{n_3}(x) = k(x^N - \alpha_1{}^N)(x^N - \alpha_2{}^N)\cdots(x^N - \alpha_{n_3}{}^N).$$

Multiplied out, this will be a polynomial in powers of x^N, the terms being of alternating sign. If we now consider the product

$$[P(x)p_{2n_1}(x)q_{n_2}(x)][S(x)s_{n_3}(x)] = P(x)S(x)f(x), \tag{3}$$

we see that since the polynomial in the first brackets is of degree N, and has all its terms positive, the product will present, first a group of positive terms (if we take out the constant factor k) beginning with that

* I here follow closely my paper "The Degree of a Cartesian Multiplier" alreadyreferred to.

in $x^{N(n_3+1)}$, then a group of negative terms beginning either with that in x^{Nn_3} or the next non-vanishing term, and so on through $n_3 + 1$ sequences of alternating sign, giving the desired n_3 variations of sign. We have thus proved that $P(x)S(x)$ is a Cartesian multiplier of $f(x)$.

The multiplier thus obtained is, of course, far from being, in general, of minimum degree. In fact, an explicit formula for m, this minimum degree, would seem very difficult to obtain, but there is considerable interest in obtaining a superior limit for it, or, what is the same thing since there are Cartesian multipliers of all degrees greater than the minimum,* obtaining an expression for the degree of some particular multiplier.

Let m_1 be the maximum degree for the multipliers of the quadratic factors of $p_{2n_1}(x)$. Then the reader will readily obtain for N the inequality

$$N \leqq m_1 n_1 + 2n_1 + n_2.$$

The degree of $P(x)$ is at most $m_1 n_1$, and that of $S(x)$ is $n_3(N-1)$, hence if M is the degree of $P(x)S(x)$ we have

$$M \leqq m_1 n_1 + n_3(N-1) \leqq m_1 n_1(1 + n_3) + n_3(2n_1 + n_2 - 1).$$

Since

$$n = 2n_1 + n_2 + n_3, \qquad n_1 > 0,$$

so that

$$n_1 \leqq \frac{n}{2}, \qquad 1 + n_3 \leqq n - 1, \qquad 2n_1 + n_2 \leqq n,$$

we have the inequality

$$(4) \quad M \leqq m_1 \frac{n(n-1)}{2} + (n-2)(n-1) = \frac{n-1}{2}[n(m_1+2) - 4].\dagger$$

We may note that this holds even when $n_1 = 0$, in which case $m = 0$. There is, then, always a Cartesian multiplier of the degree given by the last expression in (4).

It remains to replace m_1 above by an expression in terms of the coefficients.

If the pairs of imaginary roots of $f(x)$ have the form

$$\rho_k(\cos\theta_k \pm i \sin\theta_k), \qquad k = 1, 2, \cdots, n_1,$$

and if ϕ is the least of the angles θ_k, then it follows from § 4 that we can

* Obviously $x^r\phi(x)$ is a Cartesian multiplier, for all positive integral values of r, if $\phi(x)$ is a Cartesian multiplier.

† Compare the evaluation, much lower in general, in "The Degree of a Cartesian Multiplier," l. c. Both these evaluations for M are, however, usually so much larger than m that refinements here are unimportant.

take m_1 as the least integer verifying

$$(m_1 + 2)\phi \geqq \pi,$$

or, if $0 < \psi \leqq \phi$,

$$(m_1 + 2)\psi \geqq \pi, \tag{5}$$

since m_1 may assume all integral values greater than its minimum. Now by a well-known formula of algebra,* if D is the discriminant of $f(x)$ it can be expressed, provided $f(x) = 0$ has no multiple roots, as the product of $(-1)^{[n(n-1)]/2} a_0^{2n-2}$ into the product of the squares of all the differences of pairs of roots of $f(x) = 0$. These pairs are $[n(n-1)]/2$ in number. We may use a modified formula in case there are multiple roots, but since these can be removed from $f(x) = 0$ by rational processes, we shall suppose them absent. If ρ is the modulus of the conjugate pair of roots having the minimum angle, ϕ, then the square of the difference of those two roots is $(2\rho \sin \phi)^2$. If R is a superior limit for the moduli of roots of $f(x) = 0$ (for example, we may take $R = 1 + (r/r_0)$, where $r_0 = |a_0|$, and r is the greatest of the numbers $|a_0|, |a_1|, \cdots, |a_n|$), then the product of the squares of all the differences of pairs of roots other than the pair of minimum angle, ϕ, will be of modulus less than, or equal, $(2R)^{n(n-1)-2}$, since the modulus of each difference $\leqq 2R$. Thus we obtain

$$\begin{aligned} |D| &\leqq |a_0|^{2n-2}(2\rho \sin \phi)^2 (2R)^{n(n-1)-2} \\ &\leqq |a_0|^{2n-2}(2R)^{n(n-1)} \sin^2 \phi, \end{aligned}$$

or

$$\phi > \sin \phi \geqq \frac{|D|^{1/2}}{|a_0|^{n-1}(2R)^{n(n-1)/2}}. \tag{6}$$

We may now take the last expression in (6) for ψ, whence, by (5), we have

$$m_1 \geqq \frac{|a_0|^{n-1}(2R)^{[n(n-1)]/2}}{|D|^{1/2}} \cdot \pi - 2.\dagger \tag{7}$$

If this is substituted in the last expression in (4) we have a number which is the degree of some Cartesian multiplier.

6. **A geometrical configuration associated with the problem of polynomial multipliers.**‡ In § 5 we have obtained in terms of the coefficients a number which will be the degree of some Cartesian multiplier. If, now, we systematize the examination of all polynomials of this degree, we shall obtain all Cartesian multipliers of not only the degree considered but all

* Netto, "Vorlesungen über Algebra," vol. 1, p. 177.

† If the coefficients of $f(x)$ are integers we shall have $|D| \geqq 1$, so that we can use instead of (7) a simpler, but stronger inequality where $|D|^{1/2}$ is replaced by 1.

‡ Compare, throughout this section, "An Extension of Descartes's Rule of Signs," Math. Annalen, vol. 73 (1912), p. 424.

lower degrees, since if this degree is R, and there is a Cartesian multiplier $\phi(x)$ of lesser degree, r, then $x^{R-r}\phi(x)$ will be among the Cartesian multipliers of degree R. Our object, then, will be to determine what polynomials $\phi(x)$, among all those of a given degree, give to the product $\phi(x)f(x)$ the least number of variations of sign.

For simplicity, let us start with the general quadratic multiplier

$$\phi(x) = x^2 + z_1 x + z_2, \tag{1}$$

and write

$$F(x) = \phi(x)f(x) = A_0 x^{n+2} + A_1 x^{n+1} + \cdots + A_{n+2}, \tag{2}$$

where

$$\begin{aligned} A_0 &= a_0, \\ A_1 &= a_1 + a_0 z_1, \\ A_2 &= a_2 + a_1 z_1 + a_0 z_2, \\ &\cdots\cdots\cdots\cdots \\ A_n &= a_n + a_{n-1} z_1 + a_{n-2} z_2, \\ A_{n+1} &= a_n z_1 + a_{n-1} z_2, \\ A_{n+2} &= a_n z_2. \end{aligned} \tag{3}$$

We wish to choose z_1 and z_2 so that the sequence $A_0, A_1, \cdots, A_{n+2}$ presents as few variations of sign as possible.

Now if z_1 and z_2 are interpreted as Cartesian coördinates in the plane, the equations $A_i = 0$ ($i = 1, \cdots, n+2$, with the exception of values for i for which A_i is constant with respect to z_1 and z_2) will correspond to straight lines. If the point (z_1, z_2) is on the *positive side* of $A_i = 0$, these values for z_1 and z_2 make A_i positive; and A_i will be negative on the *negative side* of the line $A_i = 0$. The *A-configuration* formed by these lines will surely have at least one *vertex*, i. e., a finite intersection point, since $A_1 = 0$ and $A_2 = 0$ always intersect. The lines of the A-configuration thus divide the plane into regions of finite or infinite extent, and it is readily seen that there must be at least one vertex on the boundary of each region since every line $A_i = 0$ is met by either $A_1 = 0$ or $A_2 = 0$. We at once deduce the following property of the A-configuration:

THEOREM I. *There is at least one vertex of the A-configuration for which $\phi(x)$ gives to the product $F(x) = \phi(x)f(x)$ as few variations of sign in the sequence of its coefficients as are possible for any quadratic multiplier.*

To prove this, take any point P of the (z_1, z_2) plane, and let it move from its initial position to a vertex without crossing any line of the A-configuration. From the foregoing, this will evidently be always possible.

As the A's vary with the position of P the sign of each A is preserved, or at most certain A's may vanish, and the path of P can be so taken that each vanishing A remains equal to zero until the vertex is reached. Thus the vertex cannot give to the A's more variations than did the initial position of P.

We shall thus obtain multipliers which give to the A's the least possible number of variations of sign if we take z_1 and z_2 for each vertex of the A-configuration and use in (1) those values which give to the A's the *least* number of variations of sign. There are at most $[(n+1)(n+2)]/2$ vertices, so that the solution of our problem involves the trial of no more than this number of multipliers.* In practice this work is much facilitated by a diagram for the A-configuration in which the positive and the negative side of each line is indicated by a corresponding + or − sign

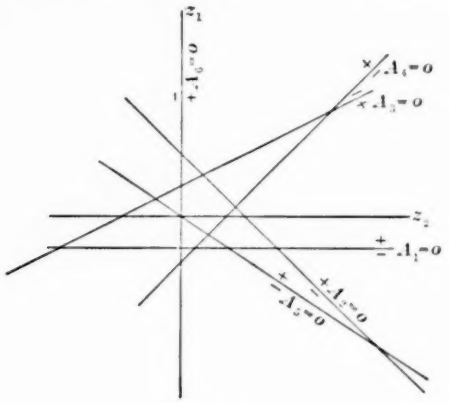

FIG. 2.

on the appropriate side. In Fig. 2 we reproduce from the Annalen article quoted a diagram for the function

$$f(x) = x^4 + x^3 - 2x^2 + 2x + 3.$$

The reader will readily verify the fact that the intersection of $A_2 = 0$ and $A_3 = 0$ is on the positive side of all the other lines, so that this is a minimizing vertex, and similarly for the intersection of $A_2 = 0$ and

* As a matter of fact, we may at once reject vertices for which $\phi(x) = 0$ has a real positive root, since, if x_1 and x_2 are positive the product $(x - x_1)(x - x_2)f(x)$ has at least two more variations of sign than $f(x)$ itself, i. e., than are given by the multiplier x^2, while a product $(x - x_1)(x + x_2)f(x)$ has at least one more variation of sign than $(x^2 + x_2x)f(x)$. This is equivalent to saying that we need examine only vertices for which

$$z_2 \geqq 0, \qquad z_1 + 2\sqrt{z_2} > 0,$$

that is, vertices on the positive side of $z_2 = 0$ or on that line, and above the lower half of the parabola $z_1^2 = 4z_2$.

$A_i = 0$, and that of $A_3 = 0$ and $A_4 = 0$. The only Cartesian multipliers of the second degree correspond to points within or on the triangle of which these three points are the vertices. Since a multiplier of the first degree would have corresponded to a minimizing vertex on the z_1 axis, two is the least degree for a Cartesian multiplier.

The above example indicates how to attack the problem of determining all possible Cartesian multipliers of degree 2, in case any such exist. The minimizing vertices are first determined; values of z_1 and z_2 need only then be considered which are in areas or on line segments adjacent to such vertices. A single point of each such area or segment is all we need to test. The example of the function

$$f(x) = 4x^6 + 4x^5 - 2x^3 - x^2 + x + 1$$

which has but one minimizing vertex, the point $(-\frac{1}{2}, \frac{1}{2})$, shows that there are polynomials which have essentially but one Cartesian multiplier of degree 2, and none of that degree with all coefficients positive.

A more completely algebraic form may be given to these results by considering the matrix

$$(4) \qquad \begin{vmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_n & a_{n-1} & a_{n-2} \\ 0 & a_n & a_{n-1} \\ 0 & 0 & a_n \end{vmatrix}$$

formed from the coefficients of the equations $A_i = 0$. It is readily seen that the substitution of the coordinates of a vertex in the A's will make two or more of them vanish, the ratios of the others being proportional to those of three rowed determinants of the matrix (4). The A's may then be replaced by sequences of determinants

$$(5) \qquad \alpha_0, \alpha_{k_1}, \cdots, \alpha_{k_n}$$

obtained from the matrix (4) as follows: *the first row of* α_{k_i} $(i = 0, 1, \cdots, n)$ *is the* $(k_i + 1)$*th row of* (4) *while the remaining rows are those of* (4) *with the first,* $(k_1 + 1)$th, $\cdots$, $(k_n + 1)$th *rows stricken out.* In terms of the sequences (5) we may restate Theorem I as follows:

THEOREM II. *Among the sequences* (5) *there will be at least one whose members do not all vanish and which will present as few variations of sign as are possible for any A-sequence.*

The reader will verify the fact that an α_0 corresponding to a vertex of the A-configuration cannot vanish.

We can now write (2) in the form

$$(6) \quad F(x) = \phi(x)f(x) = \frac{a_0}{\alpha_0}(\alpha_0 x^{n+2} + \alpha_{k_1}x^{n+2-k_1} + \cdots + \alpha_{k_n}x^{n+2-k_n})$$

provided (z_1, z_2) is a vertex. Here the k's are integers such that

$$0 < k_1 < k_2 \cdots < k_n < n + 3.$$

It is of interest to note a simple expression for the α's in terms of the roots of $f(x) = 0$. Let these roots be $x_1, \cdots, x_n$, and for simplicity let us consider the case where there are no multiple roots. Since the roots of $f(x) = 0$ are also roots of $F(x) = 0$, we have the set of n equations to determine the ratios of the α's

$$(7) \quad \alpha_0 x_i^{n+2} + \alpha_{k_1}x_i^{n+2-k_1} + \cdots + \alpha_{k_n}x_i^{n+2-k_n} = 0 \quad (i = 1, 2, \cdots, n).$$

Let us consider, then, the matrix

$$(8) \quad \begin{vmatrix} x_1^{n+2} & x_1^{n+1} & \cdots & x_1 & 1 \\ x_2^{n+2} & x_2^{n+1} & \cdots & x_2 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_n^{n+2} & x_n^{n+1} & \cdots & x_n & 1 \end{vmatrix}.$$

If the matrix M formed of the first, k_1th, $\cdots$, k_nth columns of (8) is of rank n, then by solving equations (7) for the α's we shall obtain the latter in terms of n-rowed determinants of the matrix (M). The rank of (8) itself is, however, always n when $f(x)$ has no multiple roots, since the determinant formed of its last n columns is the product of the differences of pairs of roots of $f(x) = 0$. This enables us to see what characterizes the case where M is of rank less than n. Suppose M formed from (8) by omitting the jth and kth columns, and let the rank of M be $n_1 < n$. Then there will be $n_1 + 1$ of the equations

$$A_0 x_i^{n+2} + A_1 x_i^{n+1} + \cdots + A_{n+2} = 0, \quad i = 1, 2, \cdots, n$$

from which we can eliminate all the A's but A_j and A_k, and we shall have a relation

$$D_j A_j + D_k A_k = 0,$$

which is independent of the z's and in which D_j and D_k are not both zero, since if this were so a matrix formed of n_1 rows of (8) would be of rank less than n_1. This is impossible for the same reason that the rank of M

cannot be less than n. Thus the coefficients of A_j and A_k are proportional. These coefficients, however, form the last two rows of each of the three-rowed determinants which are the coefficients α of (7).

We thus see that *if M is of rank less than n all the coefficients in* (7) *vanish. Thus in all cases the α's are proportional to n-rowed determinants of M.* In case M is of rank n_1 less than n there will be α-sequences with $n - n_1$ vanishing terms, whose non-vanishing members can be expressed in terms of a matrix similar to M, formed for n_1 roots of $f(x) = 0$, and having $n_1 + 1$ columns.

In the case of multiple roots of $f(x) = 0$ the matrix (8) must be replaced by one in which if $x_1 = x_2 = \cdots = x_m$, the $m - 1$ rows following the first are replaced by the coefficients of $A_0, A_1, \cdots, A_{n+2}$ in the equations

$$\frac{d^i}{dx^i}[\phi(x)f(x)], \qquad i = 1, 2, \cdots, m - 1,$$

and similarly if there are other multiple roots.

The work we have carried out here in detail for the case where the multiplier, $\phi(x)$, is quadratic admits a direct extension to the case where the degree r of $\phi(x)$ is greater than two. For details the reader may consult the paper referred to at the beginning of this section. The lines $A_i = 0$ are replaced by linear $(r - 1)$-spaces, and vertices are defined as finite points in which r linearly independent members of the system $A_i = 0$ intersect. If α-sequences are defined in the same way as we have done for the case $r = 2$ we again need only examine these to find the minimum number of variations of sign possible for the coefficients of any product $\phi(x)f(x)$ where $\phi(x)$ is of degree r. *This number is a monotone decreasing function of r, and, since we have proved the existence of Cartesian multipliers, must for some number $r = R$ become, and for greater values of r remain equal to the number of positive roots of $f(x)$.*

7. **The multipliers** e^{zx}, $(x - z)^{-p}$, $(x + z)^p$. One of the first applications made by Laguerre of the validity of Descartes' rule for infinite series was a discussion of the multiplier $(x - a)^{-1}$ expanded in decreasing powers of x*, where $a > 0$. In the next section we shall give some of his results in more extended form. He remarks that repeated use of this multiplier, i. e., use of the multiplier $(x - a)^{-p}$ expanded in decreasing powers of x, produces a product series in which the number of variations of sign never increases as p becomes larger. He raises the question whether this number can ultimately be made equal to the number of roots of $f(x) = 0$ that are greater than a; by Descartes' Rule in extended form the former number is for all integral values of p greater than or equal to the latter. This question he seems to have been unable to answer

* Œuvres, vol. 1, p. 5 and following.

(l. c., p. 15). Nevertheless he succeeded in establishing a closely related result when he gave his demonstration (l. c., p. 22), which must be regarded as a mathematical classic, that e^{zx} is a Cartesian multiplier for any polynomial $f(x)$, provided z is a sufficiently large positive number. Fekete and Pólya* have since proved the similar theorem for $(x - z)^{-m}$ where n is sufficiently large, and have stated that they have established it for not only this multiplier, but also for e^{zx} and $(x + z)^m$, and have extended their results to cases where $f(x)$ is an infinite series. Their work depends on long systems of inequalities, and lacks the elegant form of Laguerre's. I have since shown† that we can establish these results for $(x - z)^{-p}$ and $(x + z)^p$ by a modified form of Laguerre's proof for e^{zx}, when $f(x)$ is a polynomial.

The multiplier e^{zx} is expressible as the limit of a modified form of $(x - z)^{-p}$. In fact we have

$$e^{zx} = \lim_{p=\infty} \left(1 - \frac{zx}{p}\right)^{-p}.$$

Descartes' Rule for infinite series shows that the number of variations of sign in the coefficients of $e^{zx}f(x)$, expanded in ascending powers of x, is greater than or equal to the number of positive roots of $f(x) = 0$. If V_z is the former number for a particular value of z, and m the latter, Laguerre shows, in somewhat incomplete form, that $V_z - m$ decreases monotonically as z increases. This proof is easily completed as follows: Let k be any positive real number greater than 1, and let $z_2 = kz_1 > 0$. If we compare the products

$$F(z_1, x) = e^{z_1 x}f(x),$$

$$F(z_2, x) = e^{z_2 x}f(x),$$

we have

$$F(z_2, x) = \lim_{p=\infty} \left(1 - \frac{z_1 x}{p}\right)^{-p(k-1)} F(z_1, x).$$

We now let p approach infinity, but in such a way that $p(k - 1)$ is an integer, and consider expansions of the above functions in positive ascending powers of x. We shall show in the next section that the series in descending powers of x for $(x - z)^{-p}f(x)$, where $z > 0$ and p is a positive integer, presents a number of variations of sign that decreases monotonically as p increases. A slight modification of this proof shows that for every p considered the expression $(1 - (z_1 x/p))^{-p(k-1)}F(z_1, x)$ cannot present more variations of sign in the coefficients of its expansion in

* *Ueber ein Problem von Laguerre*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), p. 89.

† Transactions of the American Mathematical Society, vol. 16 (1915), pp. 350–360.

ascending powers of x than does $F(z_1, x)$, and the limit of the expression referred to, as $p \doteq \infty$, cannot present more variations than it does for finite p. Thus $F(z_2, x)$ cannot present more variations than $F(z_1, x)$, as was to be proved.

Laguerre's theorem may now be stated as follows:

THEOREM I. *If z is a positive number, and V_z is the number of variations of sign in the sequence of coefficients in the development of $e^{zx}f(x)$ according to ascending powers of x, while m is the number of positive roots of $f(x) = 0$, then V_z decreases monotonically as z increases, and $V_z \geqq m$. Further, there exists a number z_1 such that for all $z > z_1$ we shall have $V_z = m$.*

It remains to prove the last part of this theorem, and we will here follow Laguerre closely.

If we write

$$f(x) = a_0 + a_1x + \cdots + a_nx^n, \tag{1}$$

$$e^{zx}f(x) = A_0 + A_1x + \frac{A_2}{2!}x^2 + \cdots + \frac{A_i}{i!}x^i + \cdots, \tag{2}$$

we have for the A's the formula

$$\frac{A_i}{i!} = z^{i-n}F(i, z), \qquad i = 0, 1, 2, \cdots, \tag{3}$$

where

$$F(i, z) = a_0z^n + ia_1z^{n-1} + \cdots + i(i-1)\cdots(i-k)a_{k+1}z^{n-k-1} + \cdots + a_ni(i-1)\cdots(i-n+1). \tag{4}$$

If we substitute $1/\omega$ for z, and x/ω for i, the polynomial $F(i, z)$ becomes $(1/\omega^n)\Phi(x, \omega)$, where

$$\Phi(x, \omega) = a_0 + a_1x + a_2x(x-\omega) + \cdots + a_{k+1}x(x-\omega)\cdots(x-k\omega) + \cdots + a_nx(x-\omega)\cdots(x-\overline{n-1}\omega), \tag{5}$$

and we have

$$A_i = (i!)z^i\Phi\left(\frac{i}{z}, \frac{1}{z}\right), \qquad i = 0, 1, 2, \cdots. \tag{6}$$

Thus V_z, as defined in our theorem, is equal to the number of variations of sign in the sequence

$$\Phi(0, \omega), \ \Phi(\omega, \omega), \ \Phi(2\omega, \omega), \ \cdots.$$

But the number of changes of sign in this sequence is less than or equal to the number of positive roots of $\Phi(x, \omega)$ regarded as a polynomial in x, since each change of sign corresponds to one or more roots; i. e., if m_ω is the number of positive roots of $\Phi(x, \omega)$ as a polynomial in x, we have

$$V_z \leqq m_\omega,$$

and hence

$$m \leqq V_z \leqq m_\omega.$$

We shall now show that if ω_1 is sufficiently small, then for all positive values of ω less than ω_1 we shall have

$$m_\omega \leqq m,$$

from which it will follow that for these values of ω we must have

$$m = V_z = m_\omega,$$

where $z = 1/\omega$.

The number ω_1 may be chosen as follows, on the supposition that $a_0 \neq 0$, a condition that may always be realized by dividing out from $f(x)$ a factor x^k, in case $a_0 = 0$: Let the discriminant of $\Phi(x, \omega)$ be $\Theta(\omega)$, and let ω_1 be the smallest positive root of $\Theta(\omega) = 0$; in case there are no positive roots, ω_1 may be as large a positive number as we please. Since

$$\Phi(x, 0) \equiv f(x)$$

it follows that $m_0 = m$. Now let ω increase continuously from zero to ω_0, where $\omega_0 < \omega_1$. The roots of $\Phi(x, \omega)$ are continuous functions of ω, and, since $a_0 \neq 0$, a zero root cannot occur. Since the vanishing of $\Theta(\omega)$ is a condition for equal roots, these also are impossible when $0 < \omega < \omega_1$. Hence as ω increases from zero to ω_0 none of the roots of $\Phi(x, \omega) = 0$ that were negative for $\omega = 0$ can become positive (since none can become zero) and none which were imaginary can become real, since they would first have to coincide. Thus $\Phi(x, \omega_0) = 0$ has at most m positive roots; i. e.,

$$m_{\omega_0} \leqq m.$$

We have thus completed the proof of our theorem, which, as will readily be seen, is true even when $a_0 = 0$.

Let us now consider briefly the modifications in the above which will prove that $(x - z)^{-p}$ is, for sufficiently large positive integral values of p, a Cartesian multiplier in the sense that the number of variations of sign in the coefficients of

$$(x - z)^{-p} f(x),$$

expanded in descending powers of x, gives the exact number of roots of $f(x) = 0$ greater than the positive number z.

We write

$$(7) \qquad (x - z)^{-p} f(x) = A_{p0} x^{n-p} + A_{p1} x^{n-p-1} + A_{p2} x^{n-p-2} + \cdots,$$

and we have for the coefficients the formula, valid for $p > n$,

$$(8) \qquad A_{pj} = \frac{(p + j - n - 1)!}{(p - 1)!\, j!} z^{j-n} F(z, p, j), \qquad j = 0, 1, 2, \cdots,$$

where $F(z, p, j)$ is a polynomial in its three arguments. By the substitutions

$$\frac{j}{p} = x, \qquad \frac{1}{p} = \omega,$$

we obtain

$$\frac{1}{p^n} F(z, p, j) = \phi(z, x, \omega), \tag{9}$$

where

$$\begin{aligned}\phi(z, x, \omega) = {} & a_0(1 + x - \omega)(1 + x - 2\omega)\cdots(1 + x - n\omega)z^n + \cdots \\ & + a_k x(x - \omega)\cdots(x - \overline{k - 1}\omega)(1 + x - \overline{k + 1}\omega) \\ & \cdots(1 + x - n\omega)z^{n-k} + \cdots + a_n x(x - \omega) \\ & \cdots(x - \overline{n - 1}\omega).\end{aligned} \tag{10}$$

From (8), (9), (10) we see that the A's are of the same sign as the terms of the sequence

$$\phi(z, 0, \omega), \quad \phi(z, \omega, \omega), \quad \phi(z, 2\omega, \omega), \quad \cdots.$$

We have

$$\phi(z, x, 0) = x^n f\left(\frac{1 + x}{x} z\right),$$

so that the number M_0 of positive roots of $\phi(z, x, 0)$ as a polynomial in x is equal to M, the number of roots of $f(x)$ greater than z.

Our proof now runs closely parallel to that of Theorem I, except that ω_1 is to be taken less than $1/n$ as well as less than the least positive root in ω of the discriminant of $\phi(z, x, \omega)$, since this will insure the absence of a root $x = 0$ of $\phi(z, x, \omega)$ when ω is between zero and ω_1. The result may be stated in parallel with Theorem I as follows:

THEOREM II. *If z is a positive number, and p a positive integer, and V_{zp} is the number of variations of sign in the sequence of coefficients in the development of $(x - z)^{-p} f(x)$ according to descending powers of x, while M is the number of positive roots of $f(x) = 0$ that are greater than z, then V_{zp} decreases monotonically as p increases, and $V_{zp} \geqq M$. Further, there exists a value $p = p_1$ such that for all $p > p_1$ we shall have $V_{zp} = M$.*

If we develop $(x - z)^{-p} f(x)$ in ascending powers of x we may state an analogous theorem regarding the number of positive roots of $f(x) = 0$ less than z.

It can be proved in similar fashion that for each positive z, if p is sufficiently large, $(x + z)^p$ is a Cartesian multiplier for $f(x)$. Here, of course, the total number of positive roots of $f(x) = 0$ is given by the number of variations of sign in the coefficients of the product. We shall, however, obtain this result in another way in § 9.

8. **Repeated division by $x - z$. Cesàro means of finite order as generalized Sturm functions.*** The expansion of $(x - z)^{-p}f(x)$ for successive values of p is for some purposes best regarded as obtained by repeated division of $f(x)$ by $(x - z)$. This is a familiar proceeding in evaluating the derived functions $f'(z), f''(z), \cdots, f^{(n)}(z)$. In any numerical example synthetic division enables us to write rapidly the table of double entry formed by the coefficients A_{pj} of formulas (7) and (8), § 7, where p has the successive values 1, 2, $\cdots$. Laguerre has shown that if we consider the sequence presented by any horizontal line of this table, the number of its variations of sign is a superior limit for the number of real roots greater than z, and that this number decreases monotonically as we go down the table. He also shows (taking, for simplicity, $z = 1$) that other sequences from this table have a similar property, namely, sequences in which we proceed along the pth line of the table until we reach the $(j + 1)$th column, subject to the condition that

$$p + j \geqq n + 1,$$

and then take a path diagonally upward to the right.

To visualize these results more completely, let us here write down the table of the A's:

$$(1) \qquad \begin{array}{cccc} A_{00} & A_{01} & A_{02} & \cdots \\ A_{10} & A_{11} & A_{12} & \cdots \\ A_{20} & A_{21} & A_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}$$

Here we have added the top row to correspond to the coefficients of $f(x)$ itself, so that each row corresponds to the result of division by $(x - z)$ of the series belonging to the preceding row, these series being in descending powers of x. We thus have

$$(2) \qquad \begin{aligned} A_{0j} &= a_j \qquad (j = 0, 1, \cdots, n), \\ &= 0 \qquad (j > n), \\ A_{p0} &= A_{00} = a_0 \qquad (p = 1, 2, \cdots), \\ A_{pj} &= A_{p-1j} + zA_{pj-1} \qquad (p > 0, j > 0), \end{aligned}$$

the last relations resulting from the division algorithm. These formulas,

* Throughout this section compare Laguerre, Œuvres, vol. 1, pp. 13–22, and the author's article in the Transactions of the American Mathematical Society referred to in § 7.

used repeatedly, yield an expression for any A in terms of the first row of (1),

$$A_{pj} = \frac{p(p+1)\cdots(p+j-1)}{j!} z^j A_{00} + \frac{p(p+1)\cdots(p+j-2)}{(j-1)!} z^{j-1} A_{01} + \cdots + \frac{p(p+1)}{2!} z^2 A_{0j-2} + pz A_{0j-1} + A_{0j}, \tag{3}$$

but perhaps the simplest way to obtain (3) is to expand the product

$$(x-z)^{-p} f(x) = (x-z)^{-p} \left[\sum_{j=0}^{\infty} A_{0j} x^{n-j} \right]$$

in decreasing powers of x.

The expression (3) is easily transformed into (8) of § 7 on account of the vanishing of A_{0j} for $j > n$; in fact this formula is valid for all $p > 0$, and $p + j > n$. We thus have

$$A_{pj} = \frac{(j+p-n-1)!}{j!(p-1)!} F(z, p, j) z^{j-n} \qquad (p+j > n,\ p > 0), \tag{4}$$

$$F(z, p, j) = a_0(p+j-1)(p+j-2)\cdots(p+j-n)z^n + \cdots + a_k j(j-1)\cdots(j-k+1)(p+j-k-1)\cdots(p+j-n)z^{n-k} + \cdots + a_n j(j-1)\cdots(j-n+1).$$

This last expression is a polynomial of nth degree in j, and we easily verify the relation

$$\lim_{j=\infty} \frac{F(z, p, j)}{j^n} = f(z). \tag{5}$$

It follows that for each $p > 0$ a j_p may be determined such that if $f(z) \neq 0$, the sign of all terms A_{pj} is that of $f(z)$ for $j > j_p$. By giving to p the values $1, 2, \cdots, r$, and designating by $s - 1$ the greatest of a set of numbers $j_1, j_2, \cdots, j_r$ we obtain the following result:

THEOREM I. *If $f(z) \neq 0$, an integer $s > n$ can be determined for every integer $r > 0$ such that every term A_{pj} of table* (1) *in that part of the table from the second to the rth row and beyond the sth column has the sign of $f(z)$.*

The importance of Cesàro means* in the theory of divergent series makes it interesting to note here that the A's are expressible in terms of Cesàro means of finite order, as defined, for instance, by Bromwich (l. c.). If by $S_j^{(p-1)}$ we denote the numerator of the $(j+1)$th Cesàro mean of order $(p-1)$ for the series

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n + 0 + 0 + \cdots,$$

we have

$$A_{pj} = z^{j-n} S_j^{(p-1)} \qquad (p > 0, j \geqq 0), \tag{6}$$

* See Bromwich, Theory of Infinite Series, p. 311.

so that if $C_j^{(p-1)}$ is the Cesàro mean indicated, *any statement regarding the signs of the* A_{pj} *can be replaced by a similar statement in terms of the means* $C_j^{(p-1)}$.

From (4) we may also verify the formula

$$(7) \qquad A_{pj} = \frac{1}{(p-1)!}\frac{d^{(p-1)}}{dz^{(p-1)}}(z^{p+j-n-1}f(z)) \qquad (p > 0,\ p + j > n).$$

This means that if we consider the terms from A_{p0} to $A_{1\,p-1}$ in a diagonal of table (1), the terms of each diagonal beyond (and including) the *principal diagonal*, i. e., that which goes from A_{n+10} to A_{1n}, are of the same sign as the terms of the Budan-Fourier sequence (in reversed order) for $x^{p-n-1}f(x)$ at $x = z$. This result is, of course, well known for $p = n + 1$, i. e., for the terms of the principal diagonal.

To follow Laguerre, we should now consider sequences from (1) which proceed first horizontally, and then diagonally upward. We shall, however, extend this examination to include sequences in which progress horizontally and diagonally upward may alternate as often as we please provided we ultimately proceed horizontally in some row below the first. Since all terms in the second row beyond the nth are of the sign of $f(z)$, we see that these sequences in effect include those studied by Laguerre. I have called this extended variety *Laguerre Sequences* in the Transactions article referred to at the beginning of this section, and have there also defined one Laguerre Sequence as *enclosing* another provided for all values of j the elements of each sequence from the jth column of (1) are such that the element of the first sequence either coincides with or lies below that of the second. A similar statement is equivalent where column and row are interchanged. For a proof of the following theorem the reader is referred to the article just mentioned:

THEOREM II. *In case* $f(z) \neq 0$, *if* L_2 *is any Laguerre Sequence enclosing* L_1, *and* V_2, V_1 *are their respective numbers of variations of sign, then* $V_1 - V_2$ *is zero or an even positive integer.*

In § 7 we have proved the existence of Laguerre Sequences, namely those corresponding to rows sufficiently far down in the table, for which $V = M$, where M is the number of real roots of $f(x) = 0$ greater than z. This gives us the following corollary of Theorem II:

THEOREM III. *For every Laguerre Sequence* $V - M = 0$ *or a positive even integer, and there exist Laguerre Sequences for which* $V = M$.

Suppose, now, that the kth horizontal row has the property that $V = M$. We may replace this sequence by one, L, which coincides with it until we have entered that part of (1) where all terms have the same sign as $f(z)$ and then proceeds diagonally upward, and we shall still have

$V = M$. Finally, since L is enclosed by the sequence belonging to the whole diagonal of which part was used in L, we have established the following result:

THEOREM IV. *In case $f(z) \neq 0$, if p is sufficiently large the diagonal sequence proceeding from A_{p+10} to A_{1p} has exactly as many variations of sign as $f(z) = 0$ has positive roots greater than z.*

A reference to § 2 and (7) enables us to state this in the alternative form: *If $f(z) \neq 0$, the Budan-Fourier functions*

$$\frac{d^i}{dx^i} x^k f(x) \qquad (i = 0, 1, \cdots, k + n)$$

are generalized Sturm functions for the interval (z, ∞), provided k is sufficiently large, z being positive.

This result, stated in terms of Cesàro means, justifies the heading of this section.

Two other theorems we will here quote from the paper referred to at the beginning of this section, where the reader will find proofs.

THEOREM V. *There exists a Laguerre Sequence for which $V = M$, and such that all enclosing sequences have the same property, while all other Laguerre Sequences have $V > M$.*

THEOREM VI. *For each positive $z = z_0$ such that $f(z_0) \neq 0$ there exists a positive integer r such that every Laguerre Sequence beginning in a row of (1) below the rth has for every $z \geqq z_0$ not a root of $f(x) = 0$ exactly as many variations of sign as $f(x)$ has roots greater than z.*

We have so far supposed that z was not a root of $f(x) = 0$. Only slight modifications are necessary to include this case; the reader may find this an interesting exercise.

9. **Repeated multiplication by $x + z$.** It may be verified that most of the results of the preceding section are true even when the coefficients of $f(x)$ are functions of z. In particular, let the polynomial $f(x-z)$ be written as a polynomial in x, and form the corresponding table (1). We designate its elements by $\bar{A}_{pj}$. Formula (7) of § 8 becomes

$$(1) \qquad \bar{A}_{pj} = \frac{1}{(p-1)!} \frac{d^{(p-1)}}{dx^{(p-1)}} [x^{p+j-n-1} f(x-z)]\Big|_{x=z} \qquad (p > 0,\ p + j > n).$$

But if we put $x - z = X$, (1) becomes

$$(2) \qquad \bar{A}_{pj} = \frac{1}{(p-1)!} \frac{d^{(p-1)}}{dX^{(p-1)}} [(X+z)^{p+j-n-1} f(X)]\Big|_{X=0},$$

so that the $\bar{A}$'s for which $p + j = m + n + 1$, where m is a positive integer, are the coefficients in the polynomial

$$(x + z)^m f(x)$$

arranged according to powers of x. By Theorem IV, since these $\bar{A}$'s are in a diagonal sequence of their table, we have only to take m sufficiently large to make the number of variations of sign in this sequence equal to the number of roots of $f(x - z) = 0$ greater than z; i. e., the number of positive roots of $f(x) = 0$. We have thus proved that *if m is a sufficiently large positive integer, $(x + z)^m$ is a Cartesian Multiplier of $f(x)$, z being any positive number.*

It would be possible to form a multiplication table for $x + z$ analogous to the division table of § 8 for $x - z$. We could prove many properties analogous to those developed in § 8, independently. This we recommend to the reader, who may then deduce results in § 8 from these.

10. **Quadratic divisors.** In § 1 we noted that Descartes' Rule applies to Laurent's series (i. e., series with an infinite number of negative as well as positive integral powers of x) having a convergence interval (a, b) where a and b are both positive, the number of variations of sign, V, in such a series for $F(x)$ being equal to M, the number of roots of $F(x) = 0$ in (a, b), or greater than M; in the latter case $V - M$ is even, under suitable hypotheses regarding divergence at a and b. This suggests at once that we shall obtain a superior limit for the roots of a polynomial $f(x) = 0$ which lie between two positive numbers a and b (we may have $a = 0$) such that $f(a) \neq 0, f(b) \neq 0$, if we expand

$$\frac{f(x)}{(x - a)(x - b)}$$

in a Laurent's series convergent in (a, b), and count the variations of sign.* We have here a substitute for the Budan-Fourier method which is apt to be easier to carry out on a numerical example.

Laguerre gives an example in which this new method gives a result more exact than the old. We reproduce this here. The problem is to find how many roots of

$$f(x) = x^5 - 5x^4 - 16x^3 + 12x^2 - 9x - 5 = 0$$

lie between 0 and 1. The Budan-Fourier Rule gives this number as either zero or two. By ordinary division we have

$$\frac{f(x)}{x(x - 1)} = x^3 - 4x^2 - 20x - 8 - \frac{17x + 5}{x(x - 1)}$$

$$= x^3 - 4x^2 - 20x - 8 - \frac{22}{x - 1} + \frac{5}{x}$$

* Laguerre, Œuvres, vol. 1, pp. 76–80.

$$= \frac{5}{x} - 8 - 20x - 4x^2 + x^3 + 22(1 + x + x^2 + x^3 + x^4 + \cdots)$$

$$= \frac{5}{x} + 14 + 2x + 18x^2 + 23x^3 + 22(x^4 + x^5 + \cdots).$$

Since there are no variations of sign in this last development, the equation $f(x) = 0$ has no roots between 0 and 1.

In general we have

$$\psi(x) = \frac{f(x)}{(x-a)(x-b)} = \phi(x) + \frac{Mx + N}{(x-a)(x-b)} = \phi(x) + \frac{A}{x-a} + \frac{B}{x-b}, \tag{1}$$

where $\phi(x)$ is a polynomial of degree $n - 2$, if $f(x)$ is of degree n, and

$$A = -\frac{f(a)}{b-a}, \qquad B = \frac{f(b)}{b-a}. \tag{2}$$

We expand $A/(x-a)$ in descending powers of x, and $B/(x-b)$ in ascending powers. If we note that all terms of the former series have coefficients of the same sign as A, while all coefficients in the latter are of opposite sign to B, we see that the only terms in the Laurent's expansion of $[f(x)]\,[(x-a)(x-b)]$ that need be considered are those from the (-1)th to the $(n-1)$th power of x.

The formulas of the preceding paragraph indicate that it may be more convenient to substitute for $\psi(x)$ in (1) the function

$$\Psi(x) = (a-b)\,\frac{xf(x)}{(x-a)(x-b)}. \tag{3}$$

Here it will be only the terms from the constant term to that in x^n that present variations of sign. These form a polynomial $F_n(x)$ whose variations of sign give a superior limit for the roots of $f(x) = 0$ between a and b. Using (1) and (2) above we obtain the formulas

$$\Psi(x) = (a-b)\,x\phi(x) + f(b)\left(\frac{x}{b} + \frac{x^2}{b^2} + \cdots + \frac{x^n}{b^n}\right) + f(a) + f(a)\,\frac{a}{x-a} + f(b)\,\frac{x^{n+1}}{b^n}\cdot\frac{1}{b-x}$$

$$= F_n(x) + \frac{af(a)}{x-a} + \frac{f(b)}{b^n}\cdot\frac{x^{n+1}}{b-x},$$

from which, in connection with (3), we may write

$$F_n(x) = \frac{(a-b)xf(x)}{(x-a)(x-b)} - \frac{af(a)}{x-a} - \frac{f(b)}{b^n}\cdot\frac{x^{n+1}}{b-x}. \tag{4}$$

But

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} - \frac{b}{x-b},$$

so that (4) may be written

$$(5) \qquad F_n(x) = a\frac{f(x)-f(a)}{x-a} + \frac{1}{b^n}\cdot\frac{x^{n+1}f(b) - b^{n+1}f(x)}{x-b}.$$

This, with changes in notation, is the formula given by Laguerre at the beginning of his paper "Sur Quelques Points de la Théorie des Equations Numériques."* Here the reader will find a most interesting treatment of this subject, to which our present paper must serve merely as an introduction.

As regards divisors which are powers of a quadratic, we will make but one remark here. Formula (3) may be written

$$\Psi(x) = \frac{[a(x-b) - b(x-a)]f(x)}{(x-a)(x-b)} = \frac{af(x)}{x-a} - \frac{bf(x)}{x-b}.$$

This suggests a similar use of a function

$$\Theta(x) = \frac{[\lambda(x-b)^p + (-1)^p\mu(x-a)^p]f(x)}{(x-a)^p(x-b)^p} = \frac{\lambda f(x)}{(x-a)^p} + \frac{(-1)^p\mu f(x)}{(x-b)^p},$$

where λ and μ are positive numbers, and p is a positive integer. The question may arise whether, for all values of p sufficiently large, the number of variations of sign in the Laurent's series for $\Theta(x)$ will not be equal to the number of roots of the numerator of $\Theta(x)$, and therefore of $f(x)$, in (a, b). If, however, p is greater than n, all terms in the series with negative powers of x will be given by $[\lambda f(x)]/[(x-a)^p]$ and in fact there will be no terms in the development of this function where a positive or zero power of x occurs. Thus the number of variations of sign in the series for $\Theta(x)$ will be the sum of the number of those in the development in descending powers of $[f(x)]/[(x-a)^p]$ and the number of those in the development in ascending powers of $[f(x)]/[(x-b)^p]$. If p is sufficiently large these numbers are greater than or equal to the number of roots of $f(x)$ greater than a, and the number of positive roots of $f(x)$ less than b, respectively, so that the number of variations of sign in the series for $\Theta(x)$ will be greater than or equal to the number of positive roots of $f(x)$ plus the number of roots of $f(x)$ in (a, b).

It is thus possible, given an interval (a, b), to construct a polynomial for which $1/[(x-a)^p(x-b)^p]$ is not a Cartesian Multiplier for (a, b) for all values of p greater than some positive number.

Northwestern University,
March, 1918.

* Acta Mathematica, vol. 4 (1884), p. 97; Œuvres, vol. 1, p. 184.

A GENERAL FORM OF INTEGRAL.

By P. J. Daniell.

Introduction. The idea of an integral has been extended by Radon, Young, Riesz* and others so as to include integration with respect to a function of bounded variation. These theories are based on the fundamental properties of sets of points in a space of a finite number of dimensions. In this paper a theory is developed which is independent of the nature of the elements. They may be points in a space of a denumerable number of dimensions or curves in general or classes of events so far as the theory is concerned. It follows that, although many of the proofs given are mere translations into other language of methods already classical (particularly those due to Young), here and there, where previous proofs rested on the theory of sets of points, new methods have been devised (see, for example, theorems 3(3), 3(4), 5(1)).

Moore† has developed a theory of a similarly general nature, but restricts himself to the use of relatively uniform sequences. This concept is not used nor is it necessary in the following paper. We consider a group of elements p, which may be whatever we choose, and certain classes of functions $f(p)$ of those elements so that to every element p of the group there exists a real number $f(p)$ (which may be infinite in certain cases). To each function of a certain class there corresponds a real number $S(f)$ or $I(f)$ which is defined so as to satisfy certain conditions. $S(f)$ is a generalized Stieltjes integral, while $I(f)$ may be called the positive integral and the latter possesses correlates of nearly all the properties of the Lebesgue integral. It is shown that any I-integral is an S-integral while any S-integral is expressible as the difference of two I-integrals.

Two symbols have been taken over from symbolic logic, namely those for logical sum and logical product. The concepts involved are used extensively by Young and by the author and the symbols have been introduced to save space and to clarify the reasoning.

The reader is referred also to Hildebrandt‡ for references to an exten-

* J. Radon, Sitzungsberichte der Akademie der Wissenschaften, Wien (1913), p. 1295. W. H. Young, Proceedings of the London Mathematical Society (1914), 13, p. 109. F. Riesz, Annales de l'Ecole Normale Superieure (1914), 31, p. 9.

† E. H. Moore, Bulletin of the American Mathematical Society (1912), 18, p. 334.

‡ T. H. Hildebrandt, Bulletin of the American Mathematical Society (1917), 24, p. 117 and p. 177.

sive literature on generalized integrals. Fréchet considers a general integral but does not discuss existence theorems so completely.*

1. **Logical addition.**—$f(p)$ is said to be the logical sum of $f_1(p)$ and $f_2(p)$ if, for each p, the value of $f(p)$ is the greater of the values of $f_1(p)$, $f_2(p)$ for that p. Symbolically

$$f(p) = f_1(p) \vee f_2(p).$$

Logical product.—$f(p)$ is said to be the logical product of $f_1(p)$ and $f_2(p)$ if, for each p, the value of $f(p)$ is the less of the values of $f_1(p)$, $f_2(p)$ for that p. Symbolically

$$f(p) = f(_1p) \wedge f_2(p).$$

Then

1(1) $$f_1 \vee f_2 + f_1 \wedge f_2 = f_1 + f_2,$$

1(2) $$(-f_1) \vee (-f_2) = -(f_1 \wedge f_2).$$

It is assumed that there is an initial class T_0 of numerically valued functions $f(p)$ of the elements p where T_0 is closed with respect to the operations:—multiplication by a constant (C), addition (A), logical addition (G) and logical multiplication (G'). It is further assumed that the functions of T_0 are limited, that is corresponding to any $f(p)$, a finite number $K(f)$ can be found such that $|f(p)| < K(f)$ for all p. The following properties C, A, L, P, M are essential in what follows. $f, f_1, f_2, \cdots f_n, \cdots$ represent functions of the elements p belonging to the class T_0. Let $U(f)$ be a functional operation on f, then each of the properties is as follows:

(C) $$U(cf) = cU(f),$$

where c is constant;

(A) $$U(f_1 + f_2) = U(f_1) + U(f_2);$$

(L) If $f_1(p) \geqq f_2(p) \geqq \cdots$ and if $\lim f_n(p) = 0$ for all p, $\lim U(f_n) = 0$.

(P) If $f(p) \geqq 0$ for all p, $U(f) \geqq 0$.

(M) A functional operation $M(\varphi)$ exists for all functions of the type $|f|$, where f is of class T_0, such that if $\varphi \leqq \psi$, $M(\varphi) \leqq M(\psi)$, and such that $|U(f)| \leqq M(|f|)$. The I-integral, or $I(f)$, is a functional on functions of T_0 satisfying $(C)(A)(L)(P)$, while the S-integral satisfies $(C)(A)(L)(M)$.

The class T_0 is also restricted so as to include only such functions f that $I(f)$, $M(|f|)$ are finite for each f, though these integrals will not be bounded in their class. A few instances of the theory are as follows:

(a) The element p is a real number x in an interval (a, b). The

* M. Fréchet, Bulletin de la Société Mathématique de France (1915), 43, p. 249.

class T_0 is the class of continuous functions. Then the Riemann integral is an I-integral and the Stieltjes integral an S-integral. The extension to the class of summable functions leads to Lebesgue or Young integrals in the former case and to Radon or Young-Stieltjes integrals in the latter.

(*b*) p is a complex of numbers $(x_1, x_2, \cdots, x_n)$ or a point in a finite number of dimensions. Again T_0 is the class of continuous functions in a fundamental interval which may have infinite bounds. Again we obtain the Radon or Young-Stieltjes integral as an S-integral. The modular integral $\int f(p) \mid dv(p) \mid$ is an I-integral.

In place of continuous functions we may take T_0 to be the class of step-functions (functions constant over each of a finite number of sub-intervals), or else polygonal functions, or the class of polynomials together with their combinations by logical addition and multiplication. The resulting integral is the same. The Fréchet and Moore integrals (see references above) are also special instances.

(*c*) To show that our analysis applies also to integrals of a really new kind we may consider a particular example, namely

$$\int_0^1 f(x) d \log x.$$

Here $\log x$ is not a function of limited variation. Take T_0 as the class of functions $f(x)$ such that $f(x)/x$ is continuous. Substitute $t = -\log x$, and the integral may be defined as

$$\int_0^\infty [f(e^{-t})e^t]e^{-t}dt.$$

Since $f(x)/x$ is continuous,

$$|f(e^{-t})e^t| \leqq \text{some } K,$$

or the integral is absolutely convergent.

An example is interesting where a linear functional operation satisfies (C) (A) (P) but not (L), and therefore is *not* an instance. Suppose T_0 is the class of step functions $f(x)$ defined in the interval $0 \leqq x \leqq 1$, and let c be a number between 0 and 1, then we can define

$$U(f) = \lim_{\substack{\epsilon=0 \\ \epsilon>0}} f(c - \epsilon).$$

This $U(f)$ satisfies (C) (A) (P), but consider

$$\begin{aligned} f_n(x) &= 0, \qquad 0 \leqq x \leqq c - \frac{1}{n}, \\ &= 1, \qquad c - \frac{1}{n} < x < c, \\ &= 0, \qquad c \leqq x \leqq 1. \end{aligned}$$

Evidently $f_1 \geqq f_2 \geqq \cdots$, and $\lim f_n(x) = 0$ for all x. But $U(f_n) = 1$ for all n, or, $\lim U(f_n) = 1$ instead of 0.

2. **Relations between $S(f)$ and $I(f)$.**—In this and the next two paragraphs all functions mentioned will be of class T_0.

2(1). If $f(p) \leqq g(p)$, $I(f) \leqq I(g)$. For $g - f$ is of class T_0 and $g(p) - f(p) \geqq 0$.

$$\therefore \quad I(g) - I(f) = I(g - f) \qquad \text{[By } (C), (A)$$

$$\geqq 0 \qquad \text{[By } (P).$$

$|f| = f \vee (-f)$ must be of class T_0, then in condition (M), $I(|f|)$ satisfies the conditions to be satisfied by $M(|f|)$.

2(2). $$|I(f)| \leqq I(|f|).$$

For

$$-|f| \leqq f \leqq +|f|.$$

$$\therefore \quad -I(|f|) \leqq I(f) \leqq I(|f|) \qquad \text{By 2(1).}$$

Then $I(f)$ satisfies the condition (M) and it already satisfies (C) (A) (L), hence any I-integral is an S-integral.

3. If $f(p) \geqq 0$ for all p we define $I_1(f)$ *as the upper bound of* $S(\varphi)$ *for all functions* φ *of class* T_0 *such that* $0 \leqq \varphi \leqq f$.

This upper bound exists for

$$S(\varphi) \leqq M(|\varphi|) \qquad \text{by } (M).$$

$$M(|\varphi|) = M(\varphi) \qquad \text{for } 0 \leqq \varphi,$$

$$\leqq M(f) \qquad \text{for } \varphi \leqq f.$$

$$\therefore \quad S(\varphi) \leqq M(f).$$

$I_1(f)$ is called the positive integral associated with S.

3(1). If $f(p) \geqq 0$ for all p, $I_1(f) \geqq 0$. $0 = 0 \cdot f$ is a function of class T_0 and $S(0) = 0$. But 0 is one of the functions φ by which $I_1(f)$ is defined.

$$\therefore \quad I_1(f) \geqq 0.$$

Thus $I_1(f)$ satisfies condition (P).

3(2). If $f(p) \geqq 0$ for all p and c is a positive constant,

$$I_1(cf) = cI_1(f).$$

For if $0 \leqq \varphi \leqq f$, $0 \leqq c\varphi \leqq cf$, and vice versa. Also $S(c\varphi) = cS(\varphi)$.

3(3). If $f_1(p) \geqq 0$, $f_2(p) \geqq 0$ for all p, $I_1(f_1 + f_2) = I_1(f_1) + I_1(f_2)$. Firstly, $I_1(f_1 + f_2) \geqq I_1(f_1) + I_1(f_2)$. For if $0 \leqq \varphi_1 \leqq f_1$, $0 \leqq \varphi_2 \leqq f_2$, $0 \leqq \varphi_1 + \varphi_2 \leqq f_1 + f_2$.

$$\therefore \quad I_1(f_1 + f_2) \geqq S(\varphi_1 + \varphi_2), \quad \text{or} \quad S(\varphi_1) + S(\varphi_2).$$

But this is true however we vary φ_1, φ_2, hence

$$I_1(f_1 + f_2) \geqq I_1(f_1) + I_1(f_2).$$

Secondly,

$$I_1(f_1 + f_2) \leqq I_1(f_1) + I_1(f_2).$$

Let φ be any function such that $0 \leqq \varphi \leqq f_1 + f_2$, then $\varphi - f_1 \leqq f_2$,

$$\therefore \quad (\varphi - f_1) \vee 0 \leqq f_2,$$

$$\varphi = \varphi \wedge f_1 + (\varphi - f_1) \vee 0.$$

For if $\varphi(p) \geqq f_1(p)$ for a particular p, on the right-hand side the first term is f_1 and the second $\varphi - f_1$, while if $\varphi(p) \leqq f_1(p)$ the first term is φ and the second 0. Now

$$0 \leqq \varphi \wedge f_1 \leqq f_1 \qquad 0 \leqq (\varphi - f_1) \vee 0 \leqq f_2.$$

$$\therefore \quad S(\varphi \wedge f_1) \leqq I_1(f_1), \qquad S[(\varphi - f_1) \vee 0] \leqq I_1(f_2).$$

$$\therefore \quad S(\varphi) \leqq I_1(f_1) + I_1(f_2).$$

But we may vary φ in any way so long as $0 \leqq \varphi \leqq f_1 + f_2$.

$$\therefore \quad I_1(f_1 + f_2) \leqq I_1(f_1) + I_1(f_2).$$

From what has been proved firstly and secondly it follows that

$$I_1(f_1 + f_2) = I_1(f_1) + I_1(f_2).$$

3(4). If $f_1 \geqq f_2 \geqq \cdots$ and $\lim f_n = 0$ for all p, $\lim \mathrm{I}_1(f_n) = 0$. $I_1(f_n)$ is defined as the upper bound of $S(\varphi_n)$, where $0 \leqq \varphi_n \leqq f_n$. Hence given any positive e we can find φ_n $(0 \leqq \varphi_n \leqq f_n)$ such that

$$I_1(f_n) < S(\varphi_n) + 2^{-n}e.$$

Lemma. Given that

$$I_1(f_{n-1}) < S(\psi_{n-1}) + e_{n-1}, \qquad I_1(f_n) < S(\varphi_n) + 2^{-n}e,$$

and that

$$f_n \leqq f_{n-1}, \qquad 0 \leqq \psi_{n-1} \leqq f_{n-1}, \qquad 0 \leqq \varphi_n \leqq f_n.$$

Then

$$I_1(f_n) < S(\psi_{n-1} \wedge \varphi_n) + e_{n-1} + 2^{-n}e.$$

Since

$$0 \leqq \psi_{n-1} \leqq f_{n-1}, \qquad 0 \leqq \varphi_n \leqq f_n \leqq f_{n-1}, \qquad 0 \leqq (\psi_{n-1} \vee \varphi_n) \leqq f_{n-1}.$$

$$\therefore \quad S(\psi_{n-1} \vee \varphi_n) \leqq I_1(f_{n-1}) < S(\psi_{n-1}) + e_{n-1},$$

$$\psi_{n-1} \wedge \varphi_n = \psi_{n-1} + \varphi_n - (\psi_{n-1} \vee \varphi_n) \qquad \text{[By 1(1).}$$

$$\therefore \quad S(\psi_{n-1} \wedge \varphi_n) = S(\psi_{n-1}) + S(\varphi_n) - S(\psi_{n-1} \vee \varphi_n) > S(\varphi_n) - e_{n-1}.$$

$$\therefore \quad I_1(f_n) < S(\varphi_n) + 2^{-n}e < S(\psi_{n-1} \wedge \varphi_n) + e_{n-1} + 2^{-n}e,$$

which proves the lemma.

For each f_n in the sequence choose the appropriate φ_n so that

$$I_1(f_n) < S(\varphi_n) + 2^{-n}e.$$

Let

$$\psi_1 = \varphi_1, \qquad \psi_n = \psi_{n-1} \wedge \varphi_n \quad (n = 2, 3, \cdots),$$

$$e_1 = e/2, \qquad e_n = e_{n-1} + 2^{-n}e \quad (n = 2, 3, \cdots).$$

Then, using the Lemma successively with $n = 2, 3, \cdots$, we get

$$I_1(f_n) < S(\psi_n) + e_n, \quad e_n = e(1/2 + 1/4 + \cdots + 2^{-n}) < e.$$

$$\therefore \quad I_1(f_n) < S(\psi_n) + e.$$

Moreover $0 \leqq \psi_n \leqq \varphi_n \leqq f_n$, and $\lim f_n = 0$, therefore $\lim \psi_n = 0$.

$$\psi_1 \geqq \psi_2 \geqq \cdots \text{ and } \lim \psi_n = 0.$$

$$\therefore \quad \lim S(\psi_n) = 0 \qquad \text{[By } (L).$$

$$\therefore \quad \lim I_1(f_n) \leqq e.$$

But e was any positive quantity.

$$\therefore \quad \lim I_1(f_n) = 0.$$

Hence $I_1(f)$ satisfies the condition (L).

If $f = \varphi - \psi$, where $\varphi \geqq 0$, $\psi \geqq 0$, by definition

$$I_1(f) = I_1(\varphi) - I_1(\psi).$$

This definition is self-consistent, for if

$$f = \varphi_1 - \psi_1 = \varphi_2 - \psi_2, \qquad \varphi_1 + \psi_2 = \varphi_2 + \psi_1,$$

$$I_1(\varphi_1) + I_1(\psi_2) = I_1(\varphi_2) + I_1(\psi_1) \qquad \text{[By 3(3).}$$

$$\therefore \quad I_1(\varphi_1) - I_1(\psi_1) = I_1(\varphi_2) - I_1(\psi_2).$$

$I_1(f)$ satisfies condition (C). For if c is positive

$$I_1(cf) = I_1(c\varphi) - I_1(c\psi) = cI_1(\varphi) - cI_1(\psi) \qquad \text{[By 3(2)}$$

$$= cI_1(f).$$

If c is negative,

$$I_1(cf) = I_1(-c\psi) - I_1(-c\varphi)$$

$$= -cI_1(\psi) - [-cI_1(\varphi)] \qquad \text{[By 3(2)}$$

$$= cI_1(f).$$

If $c = 0$, $I_1(cf) = I_1(0) = 0 = 0 \cdot I_1(f)$.

$I_1(f)$ satisfies condition (A). For we have

$$\begin{aligned} I_1(f_1 + f_2) &= I_1(\varphi_1 + \varphi_2) - I_1(\psi_1 + \psi_2) \\ &= I_1(\varphi_1) + I_1(\varphi_2) - I_1(\psi_1) - I_1(\psi_2) \qquad [\text{By } 3(3) \\ &= I_1(f_1) + I_1(f_2). \end{aligned}$$

We have already proved that $I_1(f)$ also satisfies conditions (L) (P). Hence $I_1(f)$ is an I-integral.

4. We define further

$$I_2(f) = I_1(f) - S(f),$$

the negative integral associated with S. Also we define

$$I(f) = I_1(f) + I_2(f) = 2I_1(f) - S(f),$$

the modular integral associated with S. Evidently $I_2(f)$, $I(f)$ satisfy all the conditions for I-integrals. Another definition of the modular integral $I(f)$ when f is non-negative is the upper bound of $S(\varphi)$ for all functions φ of class T_0 such that $-f \leqq \varphi \leqq f$. For then

$$0 \leqq \varphi + f \leqq 2f, \quad S(\varphi) = S(\varphi + f) - S(f).$$

Varying φ, keeping f fixed, we see that the upper bound of $S(\varphi)$ is equal to

$$I_1(2f) - S(f) = 2I_1(f) - S(f) = I(f).$$

4(1). $|S(f)| \leqq I(|f|)$, if I is the modular integral associated with S.

For

$$\begin{aligned} |S(f)| = |I_1(f) - I_2(f)| &\leqq |I_1(f)| + |I_2(f)| \\ &\leqq I_1(|f|) + I_2(|f|)[\quad \text{By } 2(2) \\ &\text{or } I(|f|). \end{aligned}$$

It can be seen that if we extend the definitions of $I_1(f)$, $I_2(f)$ to functions of a wider class T so as still to satisfy $(C)(A)$ (L) (P), and if we define

$$S(f) = I_1(f) - I_2(f), \quad I(f) = I_1(f) + I_2(f),$$

$S(f)$ will satisfy (C) (A) (L) (M) and $I(f)$ will satisfy (C) (A) (L) (P).

5. **Extension to class T_1 for any I-integral.** If $f_1 \leqq f_2 \leqq \cdots$ is a non-decreasing sequence of functions of class T_0, $\lim f_n$ exists (if we allow $+\infty$ as a value) and we say that $\lim f_n = f$ *is of class* T_1.

Then

$$I(f_1) \leqq I(f_2) \leqq \cdots,$$

and $\lim I(f_n)$ exists (if we allow $+\infty$ as a value).

5(1). If $f_1 \leqq f_2 \leqq \cdots$ are of class T_0, and if $\lim f_n \geqq$ some function h of class T_0, $\lim I(f_n) \geqq I(h)$. Let $g_n = f_n \wedge h$. Then g_n is of class T_0 and since $\lim f_n \geqq h$, $\lim g_n = h$. Also $f_{n-1} \leqq f_n$, therefore $g_{n-1} \leqq g_n$.

$$\therefore \quad h - g_1 \geqq h - g_2 \geqq \cdots \text{ and } \lim (h - g_n) = 0.$$

$$\therefore \quad \lim I(h - g_n) = 0 \qquad \text{[By } (L).$$

$$\therefore \quad \lim I(g_n) = I(h).$$

But $f_n \geqq g_n$ or $I(f_n) \geqq I(g_n)$ [By 2(1).

$$\therefore \quad \lim I(f_n) \geqq \lim \mathrm{I}(g_n) \text{ or } I(h).$$

5(2). If $f_1 \leqq f_2 \leqq \cdots$, $g_1 \leqq g_2 \leqq \cdots$ are of class T_0, and if

$$\lim f_n \geqq \lim g_n, \text{ then } \lim I(f_n) \geqq \lim I(g_n).$$

For $\lim f_n$ is of class T_1 and $\geqq \lim g_n$, and therefore $\geqq g_n$.

$$\therefore \quad \lim I(f_n) \geqq I(g_n) \quad (n = 1, 2, \cdots) \qquad \text{[By 5(1).}$$

$$\therefore \quad \lim I(f_n) \geqq \lim I(g_n).$$

5(3). If $f_1 \leqq f_2 \leqq \cdots$, $g_1 \leqq g_2 \leqq \cdots$ are of class T_0 and if

$$\lim f_n = \lim g_n, \text{ then } \lim I(f_n) = \lim I(g_n).$$

Apply 5(2) twice.

We define $I(f) = lim\ I(f_n)$, *if* f *is of class* T_1 *and defined by the non-decreasing sequence* (f_n) *of functions of class* T_0. By 5(3) this definition will be self-consistent. By 5(1), if $f \geqq 0$, $I(f) \geqq 0$. Hence condition (P) is satisfied. Evidently conditions (A) (C) will be satisfied so long as in (C) the constant c is positive.

Note. We have allowed $\lim I(f_n)$ to be $+\infty$ and this necessitates a reconsideration of the above theorems. In 5(1), h is of class T_0 and therefore $I(h)$ is finite, or the statement will hold even if $\lim I(f_n) = +\infty$. In 5(2) the theorem must be taken to mean, in the case where either $\lim I(f_n)$ or $\lim I(g_n)$ is $+\infty$, that at least $\lim I(f_n)$ is $+\infty$. In 5(3) if either limit is $+\infty$ so is the other.

If $I(f) = \lim I(f_n)$ is finite and f is of class T_1 we say that f *is summable.*

5(4). If $f_1 \leqq f_2 \leqq \cdots$ is a nondecreasing sequence of functions of class T_1, then $\lim f_n = f$ is also of class T_1 and $I(f) = \lim I(f_n)$. For any integer r, f_r is of class T_1 and is the limit of a nondecreasing sequence of functions of class T_0,

$$f_{r,1} \leqq f_{r,2} \leqq \cdots \leqq f_{r,s} \leqq \cdots.$$

Let g_n be the logical sum of all functions $f_{r,s}$ for which $r \leqq n$, $s \leqq n$. Then g_n is of class T_0 and $g_n \leqq g_{n+1}$.

$$\therefore \quad \lim g_n \text{ is of class } T_1.$$

If $r \leqq n, f_{r,s} \leqq f_r \leqq f_n$.

$$\therefore \quad g_n \leqq f_n \qquad \text{and} \qquad I(g_n) \leqq I(f_n).$$

If $r \leqq n$, $g_n \geqq f_{r,n}$

$$\therefore \quad \lim g_n \geqq \lim f_{r,n} \text{ or } f_r.$$

$$\therefore \quad \lim g_n \geqq f_r \quad (r = 1, 2, \cdots)$$

and

$$I(\lim g_n) \geqq I(f_r).$$

$$g_n \leqq f_n, \qquad \lim g_n \leqq \lim f_n.$$

$$\lim g_n \geqq f_r.$$

$$\therefore \quad \lim g_n \geqq \lim f_n.$$

$$\therefore \quad \lim g_n = \lim f_n.$$

But $\lim g_n$ is of class T_1, therefore $\lim f_n$ is of class T_1.

$$I(f) = I(\lim g_n) \geqq I(f_r) \text{ for all } r.$$

$$\therefore \quad I(f) \geqq \lim I(f_n).$$

$$I(f) = I(\lim g_n) \leqq I(g_n).$$

But

$$I(g_n) \leqq I(f_n).$$

$$\therefore \quad I(f) \leqq \lim I(f_n).$$

$$\therefore \quad I(f) = \lim I(f_n).$$

6. **Semi-integrals.** For any function f we define $\dot{I}(f)$, *the upper semi-integral of* f, *as the lower bound of* $I(\varphi)$ *for all functions* φ *of class* T_1, *such that* $\varphi \geqq f$.

6(1). If c is a positive constant, $\dot{I}(cf) = c\dot{I}(f)$. For if $\varphi \geqq f$, $c\varphi \geqq cf$ and vice versa, and $I(c\varphi) = cI(\varphi)$.

6(2) $$\dot{I}(f_1 + f_2) \leqq \dot{I}(f_1) + \dot{I}(f_2).$$

For if φ_1, φ_2 are any functions of class T_1 such that $\varphi_1 \geqq f_1$, $\varphi_2 \geqq f_2$,

$$\varphi_1 + \varphi_2 \geqq f_1 + f_2.$$

$$\therefore \quad \dot{I}(f_1 + f_2) \leqq I(\varphi_1 + \varphi_2), \text{ or } I(\varphi_1) + I(\varphi_2).$$

Varying φ_1, φ_2 independently, we obtain the theorem.

6(3). If $f \leqq g$ for all p,

$$\dot{I}(f) \leqq \dot{I}(g).$$

For if φ is any function of class T_1 such that $\varphi \geqq g$, $\varphi \geqq f$.

$$\dot{I}(f) \leqq I(\varphi), \text{ etc.}$$

6(4). We define $I(f) = -\dot{I}(-f)$; then $I(f)$ is called *the lower semi-integral of* f, and $I(f) \leqq \dot{I}(f)$. For by 6(2)

$$0 = \dot{I}(0) = \dot{I}(f - f) \leqq \dot{I}(f) + \dot{I}(-f) \text{ or } \dot{I}(f) - I(f).$$

6(5) $$\dot{I}(f \vee g) + \dot{I}(f \wedge g) \leqq \dot{I}(f) + \dot{I}(g).$$

For if φ_1, φ_2 are any functions of class T_1 such that $\varphi_1 \geqq f$, $\varphi_2 \geqq g$.

$$\varphi_1 \vee \varphi_2 \geqq f \vee g. \qquad \varphi_1 \wedge \varphi_2 \geqq f \wedge g.$$

$$\therefore \quad \dot{I}(f \vee g) + \dot{I}(f \wedge g) \leqq I(\varphi_1 \vee \varphi_2) + I(\varphi_1 \wedge \varphi_2) \text{ or } I(\varphi_1) + I(\varphi_2).$$

For

$$\varphi_1 \vee \varphi_2 + \varphi_1 \wedge \varphi_2 = \varphi_1 + \varphi_2.$$

Varying φ_1, φ_2 independently, we obtain the theorem.

Corollary.

$$\dot{I}(|f|) - I(|f|) \leqq \dot{I}(f) - I(f).$$

For

$$|f| = f \vee (-f), \qquad -|f| = f \wedge (-f).$$

$$\therefore \quad \dot{I}(|f|) + \dot{I}(-|f|) \leqq \dot{I}(f) + \dot{I}(-f).$$

7. **Summability.** If $\dot{I}(f) = I(f) =$ finite, f *is said to be summable,* and we define

$$I(f) = \dot{I}(f) = I(f).$$

7(1). If f is summable and $f \geqq 0$ for all p, $I(f) \geqq 0$. For by 6(3), $\dot{I}(f) \geqq \dot{I}(0)$ or 0.

7(2). If c is any constant and f is summable, cf is summable and $I(cf) = cI(f)$.

If c is positive,

$$\dot{I}(cf) = c\dot{I}(f) = cI(f) \qquad \text{[By 6(1).}$$

$$-I(cf) = \dot{I}(-cf) = cI(-f) = -cI(f).$$

$$\therefore \quad \dot{I}(cf) = I(cf) = cI(f).$$

If c is negative,

$$\dot{I}(cf) = -c\dot{I}(-f) \qquad \text{[By 6(1)}$$

$$= cI(f)$$

$$-I(cf) = \dot{I}(-cf) = -c\dot{I}(f). \quad \therefore \quad \text{etc.}$$

7(3). If f_1, f_2 are summable, so is $f_1 + f_2$ and $I(f_1 + f_2) = I(f_1) + I(f_2)$.

For

$$\dot{I}(f_1 + f_2) \leqq \dot{I}(f_1) + \dot{I}(f_2) \qquad \text{[By 6(2),}$$

$$\text{or } I(f_1) + I(f_2).$$

$$-\underline{I}(f_1+f_2) = \dot{I}(-f_1-f_2) \leqq \dot{I}(-f_1) + \dot{I}(-f_2).$$

$$\text{or } -I(f_1) - I(f_2).$$

$$\therefore \quad \underline{I}(f_1+f_2) \geqq I(f_1) + I(f_2)$$

But

$$\dot{I}(f_1+f_2) \geqq \underline{I}(f_1+f_2). \quad \therefore \quad \text{etc.}$$

7(4). If f is summable, so is $|f|$ and $|I(f)| \leqq I(|f|)$. By 6(5) Cor.

$$\dot{I}(|f|) - \underline{I}(|f|) \leqq \dot{I}(f) - \underline{I}(f) = 0.$$

But $\dot{I}(|f|) \geqq \underline{I}(|f|)$. Hence the first part follows. Moreover,

$$-|f| \leqq f \leqq |f|.$$

$$\therefore \quad -I(|f|) = \dot{I}(-|f|)$$

$$\leqq \dot{I}(f) \text{ or } I(f)$$

$$\leqq \dot{I}(|f|) \text{ or } I(|f|).$$

The second part follows immediately.

7(5). If f_1, f_2 are summable, so are $f_1 \vee f_2, f_1 \wedge f_2$. For by 6(5)

$$\dot{I}(f_1 \vee f_2) + \dot{I}(f_1 \wedge f_2) \leqq \dot{I}(f_1) + \dot{I}(f_2), \text{ or } I(f_1) + I(f_2).$$

If we replace f_1 by $-f_1, f_2$ by $-f_2$, we shall replace $f_1 \vee f_2$ by $-(f_1 \wedge f_2)$ and $f_1 \wedge f_2$ by $-(f_1 \vee f_2)$.

$$\therefore \quad \dot{I}[-(f_1 \wedge f_2)] + \dot{I}[-(f_1 \vee f_2)] \leqq \dot{I}(-f_1) + \dot{I}(-f_2),$$

$$\therefore \quad \underline{I}(f_1 \wedge f_2) + \underline{I}(f_1 \vee f_2) \geqq \underline{I}(f_1) + \underline{I}(f_2), \text{ or } I(f_1) + I(f_2).$$

$$\therefore \quad \dot{I}(f_1 \vee f_2) - \underline{I}(f_1 \vee f_2) + \dot{I}(f_1 \wedge f_2) - \underline{I}(f_1 \wedge f_2) \leqq 0.$$

But each of these differences is non-negative, therefore they are both zero.

7(6). If $f_1 \leqq f_2 \leqq \cdots$ is a nondecreasing sequence of summable functions, and if $\lim I(f_n)$ is finite, $\lim f_n = f$ is summable, and $I(f) = \lim I(f_n)$; while if $\lim I(f_n) = +\infty$, $\underline{I}(f) = +\infty$.

For $-f \leqq -f_n$.

$$\therefore \quad I(-f) \leqq I(-f_n) \qquad \text{[By 6(3).}$$

$$\underline{I}(f) \geqq I(f_n) \quad (n = 1, 2, \cdots).$$

$$\therefore \quad \underline{I}(f) \geqq \lim I(f_n).$$

This proves the last part of the theorem. Given any positive e, we can choose $\varphi_1, \varphi_2, \cdots$ of class T_1 such that

$$\varphi_1 \geqq f_1, \qquad \varphi_2 \geqq f_2 - f_1, \qquad \varphi_3 \geqq f_3 - f_2, \quad \cdots,$$

and so that

$$I(\varphi_1) < I(f_1) + e/2,$$
$$I(\varphi_2) < I(f_2 - f_1) + e/4,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If $n \geqq 2$, $\varphi_n \geqq f_n - f_{n-1} \geqq 0$. We define $\psi_n = \varphi_1 + \varphi_2 + \cdots + \varphi_n$. Then ψ_n is of class T_1 and $\psi_1 \leqq \psi_2 \leqq \cdots$. By 5(4), $\lim \psi_n$ is of class T_1 and $I(\lim \psi_n) = \lim I(\psi_n)$. $\psi_n \geqq f_n$, or $\lim \psi_n \geqq \lim f_n$ or f.

$$\therefore \quad \dot{I}(f) \leqq I(\lim \psi_n) \qquad \text{[By 6(3),}$$
$$\text{or } \lim I(\psi_n).$$

$$\begin{aligned} I(\psi_n) &= I(\varphi_1) + I(\varphi_2) + \cdots + I(\varphi_n) \\ &< I(f_1) + I(f_2 - f_1) + \cdots + I(f_n - f_{n-1}) \\ &\quad + e(\tfrac{1}{2} + \tfrac{1}{4} + \cdots + 2^{-n}) \\ &< I(f_n) + e. \end{aligned}$$

$$\therefore \quad \lim I(\psi_n) \leqq \lim I(f_n) + e.$$

$$\therefore \quad \dot{I}(f) \leqq \lim I(f_n) + e.$$

But e is any positive quantity.

$$\therefore \quad \dot{I}(f) \leqq \lim I(f_n).$$

We have already shown that $\underset{\cdot}{I}(f) \geqq \lim I(f_n)$.

$$\therefore \quad \dot{I}(f) = \underset{\cdot}{I}(f) = \lim I(f_n).$$

$\therefore$ if $\lim I(f_n)$ exists, f is summable and $I(f) = \lim I(f_n)$.

7(7). If $f_1, f_2, \cdots$ is a sequence of summable functions with limit f, and if a summable function φ exists such that $|f_n| \leqq \varphi$ for all n, f is summable, $\lim I(f_n)$ exists and $= I(f)$. We must recall the method whereby the limit of a sequence is obtained. Let $g_{r,s}$ be the logical sum of $f_r, f_{r+1}, \cdots, f_{r+1}$; then $g_{r,s} \leqq g_{r,s+1} \leqq \cdots$ with limit g_r. Then $g_r \geqq g_{r+1} \geqq \cdots$, and $\lim g_r = f$, if $\lim f_n = f$. Similarly we let $h_{r,s}$ be the logical product of $f_r, f_{r+1}, \cdots f_{r+s}$, and then $h_{r,s} \geqq h_{r,s+1} \geqq \cdots$ with limit h_r. Then $h_r \leqq h_{r+1} \leqq \cdots$, and $\lim h_r = f$ if $\lim f_n = f$. f_n is summable for all n, therefore by 7(5), $g_{r,s}$ is summable. Since $f_n \leqq \varphi$, $g_{r,s} \leqq \varphi_1$. $\therefore I(g_{r,s}) \leqq I(\varphi)$. Therefore g_r is summable by 7(6). Again $g_{r,s} \geqq -\varphi$ or $-g_{r,s} \leqq \varphi$. $\therefore -g_r \leqq \varphi$. $\therefore I(-g_r) \leqq I(\varphi)$. But

$$-g_r \leqq -g_{r+1} \leqq \cdots$$

with limit $-f$. Therefore $-f$ is summable by 7(6) and

$$I(-f) = \lim I(-g_r).$$

Therefore by 7(2), f is summable and $I(f) = \lim I(g_r)$. Given any positive e we can find r_1, so that

$$I(g_r) < I(f) + e \quad (r \geqq r_1).$$

Now

$$f_r \leqq g_{r,s} \leqq g_r.$$

$$\therefore \quad I(f_r) \leqq I(g_r)$$

$$< I(f) + e \quad (r \geqq r_1).$$

Similarly we can prove that h_r is summable and $I(f) = \lim I(h_r)$. We can find r_2 so that

$$I(h_r) > I(f) - e \quad (r \geqq r_2).$$

$$f_r \geqq h_{r,s} \geqq h_r.$$

$$\therefore \quad I(f_r) \geqq I(h_r)$$

$$> I(f) - e \quad (r \geqq r_2).$$

Therefore if r_0 is the greater of r_1 and r_2,

$$|I(f_r) - I(f)| < e \quad (r \geqq r_0).$$

Hence $\lim I(f_n)$ exists and equals $I(f)$. From these theorems it follows that $I(f)$ satisfies the conditions (C) (A) (L) (P), where the functions now belong to the class of summable functions.

8. **S-integrals.** Associated with any S-integral $S(f)$ for functions of class T_0 we have three I-integrals, namely $I_1(f)$, $I_2(f)$, $I(f)$, such that

$$S(f) = I_1(f) - I_2(f), \quad I(f) = I_1(f) + I_2(f).$$

If we extend our definitions of the I-integrals to functions of class T_1, we shall still have

$$I(f) = I_1(f) + I_2(f).$$

8(1). If f is any function,

$$\dot{I}_1(f) + \dot{I}_2(f) = \dot{I}(f).$$

For if φ is any function of class T_1 such that $\varphi \geqq f$,

$$\dot{I}_1(f) + \dot{I}_2(f) \leqq I_1(\varphi) + I_2(\varphi), \text{ or } I(\varphi).$$

Varying φ, we obtain

$$\dot{I}_1(f) + \dot{I}_2(f) \leqq \dot{I}(f).$$

Again given any positive e we can choose φ_1, φ_2 of class T_1 so that $\varphi_1 \geqq f$, $\varphi_2 \geqq f$ and $\dot{I}_1(f) > I_1(\varphi_1) - \frac{1}{2}e$, $\dot{I}_2(f) > I_2(\varphi_2) - \frac{1}{2}e$. Let $\psi = \varphi_1 \wedge \varphi_2$, then $\psi \leqq \varphi_1$ and $\psi \leqq \varphi_2$.

$$\therefore \quad I_1(\varphi_1) + I_2(\varphi_2) \geqq I_1(\psi) + I_2(\psi), \text{ or } I(\psi).$$

$$\therefore \quad \dot{I}_1(f) + \dot{I}_2(f) > I(\psi) - e.$$

But $\psi \geqq f$. $\therefore \quad I(\psi) \geqq \dot{I}(f)$.

$$\therefore \quad \dot{I}_1(f) + \dot{I}_2(f) > \dot{I}(f) - e.$$

This is true for any positive e.

$$\therefore \quad \dot{I}_1(f) + \dot{I}_2(f) \geqq \dot{I}(f).$$

But we have already proved that

$$\dot{I}_1(f) + \dot{I}_2(f) \leqq \dot{I}(f). \quad \therefore \text{ etc.}$$

8(2). If f is summable (I), it is summable (I_1) and (I_2), and

$$I(f) = I_1(f) + I_2(f).$$

For

$$\dot{I}_1(f) + \dot{I}_2(f) = \dot{I}(f) = I(f).$$

$$\dot{I}_1(-f) + \dot{I}_2(-f) = \dot{I}(-f) = -I(f).$$

$$\therefore \quad I_1(f) + I_2(f) = I(f).$$

$$\therefore \quad \dot{I}_1(f) - I_1(f) + \dot{I}_2(f) - I_2(f) = 0.$$

But each of these differences is non-negative, therefore each must be zero separately. Then f is summable (I_1) and (I_2) and

$$I(f) = \dot{I}_1(f) + \dot{I}_2(f) = I_1(f) + I_2(f).$$

f is said to be summable (S) if, and only if, it is summable (I), where I is the modular integral associated with S. Hence if f is summable (S), it is summable (I_1) and (I_2) by 8(2).

We define

$$S(f) = I_1(f) - I_2(f).$$

Then $S(f)$ satisfies all the conditions (C) (A) (L) (M) for functions summable (S).

Many of the theorems already obtained for the I-integral can be immediately stated also for the S-integral.

Thus 7(2, 3, 5, 7) are true if we replace I everywhere by S. 7(4) becomes:

If f is summable (S), so is $|f|$, and

$$|S(f)| \leqq I(|f|),$$

where I is the modular integral associated with S. 7(6) becomes:

If $f_1 \leqq f_2 \leqq \cdots$ are summable (S) and if $\lim I(f_n)$ exists, where I is

the associate modular integral, then $\lim f_n = f$ is summable (S) and

$$S(f) = \lim S(f_n).$$

8(3). The necessary and sufficient condition that f be summable (S) is that given any positive e there exists a function f_e of class T_0 such that

$$\dot{I}(|f - f_e|) < e,$$

where I is the modular integral associated with S.

Also in this case

$$S(f) = \lim_{e \doteq 0} S(f_e).$$

The condition is sufficient for

$$f = f_e + f - f_e \leqq f_e + |f - f_e|.$$

$$\therefore \quad \dot{I}(f) \leqq I(f_e) + \dot{I}(|f - f_e|) < I(f_e) + e.$$

Similarly

$$\dot{I}(-f) < I(-f_e) + e,$$

$$\underset{\cdot}{I}(f) > I(f_e) - e.$$

$$\therefore \quad \dot{I}(f) - \underset{\cdot}{I}(f) < 2e.$$

This is true for any positive e, therefore $\dot{I}(f) = \underset{\cdot}{I}(f)$. They are also finite for $I(f_e)$ and e are finite. Also

$$|S(f) - S(f_e)| = |S(f - f_e)| \leqq I(|f - f_e|) < e.$$

$$\therefore \quad S(f) = \lim S(f_e).$$

The condition is necessary, for given any positive e there exists a summable function φ of class T_1 such that $\varphi \geqq f$ and

$$\dot{I}(f) \leqq I(\varphi) < \dot{I}(f) + \tfrac{1}{2}e.$$

$$\dot{I}(|\varphi - f|) = \dot{I}(\varphi - f)$$

(since $\varphi \geqq f$)

$$\leqq I(\varphi) + \dot{I}(-f), \quad \text{or} \quad I(\varphi) - \underset{\cdot}{I}(f), \quad \text{or} \quad \dot{I}(\varphi) - \dot{I}(f),$$

for f is summable.

$$\therefore \quad \dot{I}(|\varphi - f|) < \tfrac{1}{2}e.$$

φ is summable and of class T_1, therefore there exists a function f_e of class T_0 such that $f_e \leqq \varphi$ and

$$I(f_e) > I(\varphi) - \tfrac{1}{2}e.$$

Then

$$I(|\varphi - f_e|) < \tfrac{1}{2}e.$$

$$\therefore \quad I(|f - f_e|) \leqq \dot{I}(|\varphi - f| + |\varphi - f_e|)$$

$$\leqq \dot{I}(|\varphi - f|) + \dot{I}(|\varphi - f_e|) < e.$$

Evidently the condition for summability, or the class of summable functions depends both on the operation S and on the class T_0.

9. **Measure.** It is usual, though not necessary, to define the integral in terms of the measure of certain fundamental sets. Let us suppose that the measures of a certain class of elementary, or initial sets, or collections E, of the p are given. In connection with a collection E we can define a function $= 1$ when p belongs to E, $= 0$ otherwise. We can agree to call the measure of E, the integral of the corresponding function. The class T_0 is then taken as the class of all functions which are linear combinations of these elementary set-functions. It will then be closed with respect to the operations (C) (A). For any set E whatever we can say that it is measurable if the corresponding function is summable, and we can identify its measure with the integral of that function. This question requires however a separate and careful consideration. The author wishes to point out, without proof, a simple manner in which the Stieltjes integral can be generalized.

In the ordinary Stieltjes integral $\int_a^b f(x)d\alpha(x)$, $f(x)$ is a continuous function $(a \leqq x \leqq b)$ and $\alpha(x)$ of limited variation, that is such that

$$\sum_{i=1}^{n} | \alpha(x_i) - \alpha(x_{i-1}) | \leqq M,$$

for all subdivisions $a = x_0 < x_1 < \cdots < x_n = b$. Suppose that $\gamma(x)$ is not a function of limited variation but that

$$\sum_{i=1}^{n} x_i | \gamma(x_i) - \gamma(x_{i-1}) | \leqq M,$$

for all subdivisions $0 = x_0 < x_1 < \cdots < x_n = 1$. Then if $f(x)/x$ is continuous $(0 \leqq x \leqq 1)$, we can define

$$\int_0^1 f(x)d\gamma(x) = \int_0^1 \frac{f(x)}{x} xd\gamma(x).$$

Of course this is only a transformation of the integral, $\int_0^1 \frac{f(x)}{x} d\alpha(x)$, where $\alpha(x) = \int_0^x xd\gamma(x)$ is of limited variation. It shows nevertheless that so long as suitable restrictions are placed on the integrand, integrals similar to that of Stieltjes can be defined with respect to functions which are not of limited variation.

RICE INSTITUTE,
HOUSTON, TEXAS.

ELASTIC STRESSES IN AN INFINITE SOLID WITH A SPHERICAL CAVITY.

By T. H. Gronwall.

Sir Joseph Larmor has investigated the stresses in a solid having a cavity of dimensions very much smaller than those of the solid.* By a general physical argument, making no use of the mathematical expressions for the stress components, he arrives at the conclusion (l. c., p. 72) that the presence of the cavity materially affects the shearing stresses only, while in a column, for instance, acted upon by tensile or compressive external forces only, the tensile or compressive stresses are increased very slightly, viz., in the ratio of the cross section area of the cavity to that of the column. As will be shown below, this conclusion in respect to the tensile or compressive stresses is not borne out by the mathematical theory of elasticity.

Let us consider a solid extending to infinity in all directions, and having a spherical cavity of radius a with its center at the origin. Two cases of stress distribution at infinity are of special interest:

I. Pure shear:

$$X_x = Y_y = Z_z = Y_z = Z_x = 0,$$

$$X_y = S = \text{const., for } r^2 = x^2 + y^2 + z^2 = \infty;$$

II. Pure tension or compression:

$$X_x = Y_y = X_y = Y_z = Z_x = 0, \qquad Z_z = T = \text{const., for } r^2 = \infty.$$

The former case has been treated completely by Professor Love,† who derives the displacements u, v, w by the method of Lord Kelvin‡ and finds that at the surface of the cavity, for $x = y = 0, z = a$,

$$X_y = \frac{15\lambda + 30\mu}{9\lambda + 14\mu} S,$$

which is nearly twice as large as S.

* The influence of flaws and air-cavities on the strength of materials, Philosophical Magazine, ser. 5, vol. 33 (1892), pp. 70–78.

† Larmor, l. c., pp. 76–78, and Love's Theory of Elasticity, 2d ed., § 178.

‡ Phil. Trans. Roy. Soc., vol. 153 (1863); Math. and Phys. Papers, vol. 3 (Cambridge, 1890), p. 351.

In the case of pure tension or compression at infinity, I have applied the same method, obtaining the following values of the displacements:

$$u = \frac{a^3 T}{2\mu}\left[-\frac{\lambda+6\mu}{2(9\lambda+14\mu)}\frac{x}{r^3} + \frac{\lambda+\mu}{9\lambda+14\mu}(a^2-r^2)\frac{\partial}{\partial x}\frac{3z^2-r^2}{r^5} - \frac{\lambda}{3\lambda+2\mu}\frac{x}{a^3}\right],$$

$$v = \frac{a^3 T}{2\mu}\left[-\frac{\lambda+6\mu}{2(9\lambda+14\mu)}\frac{y}{r^3} + \frac{\lambda+\mu}{9\lambda+14\mu}(a^2-r^2)\frac{\partial}{\partial y}\frac{3z^2-r^2}{r^5} - \frac{\lambda}{3\lambda+2\mu}\frac{y}{a^3}\right],$$

$$w = \frac{a^3 T}{2\mu}\left[\frac{11\lambda+26\mu}{2(9\lambda+14\mu)}\frac{z}{r^3} + \frac{\lambda+\mu}{9\lambda+14\mu}(a^2-r^2)\frac{\partial}{\partial z}\frac{3z^2-r^2}{r^5} + \frac{2\lambda+2\mu}{3\lambda+2\mu}\frac{z}{a^3}\right].$$

The stress distribution is evidently one of revolution about the z-axis; calculating the stress components for $x = a$, $y = z = 0$, we find

$$X_x = 0, \qquad Y_y = -\frac{11\lambda+6\mu}{18\lambda+28\mu}T, \qquad Z_z = \frac{39\lambda+54\mu}{18\lambda+28\mu}T,$$
$$X_y = Y_z = Z_x = 0.$$

Thus at any point on the intersection of the sphere with the xy-plane, the stress Z_z is about twice as large as at infinity. This is in entire qualitative accordance with the result obtained by Kirsch in the corresponding plane problem: the presence of a circular hole in a plate of infinite dimensions makes the maximum tensile stress three times its value at infinity.

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